ASTR 620: Planetary Processes
Professor Eric Nielsen

Lecture 4: Resonances, Lagrange Points, Roche Limit

Logistics

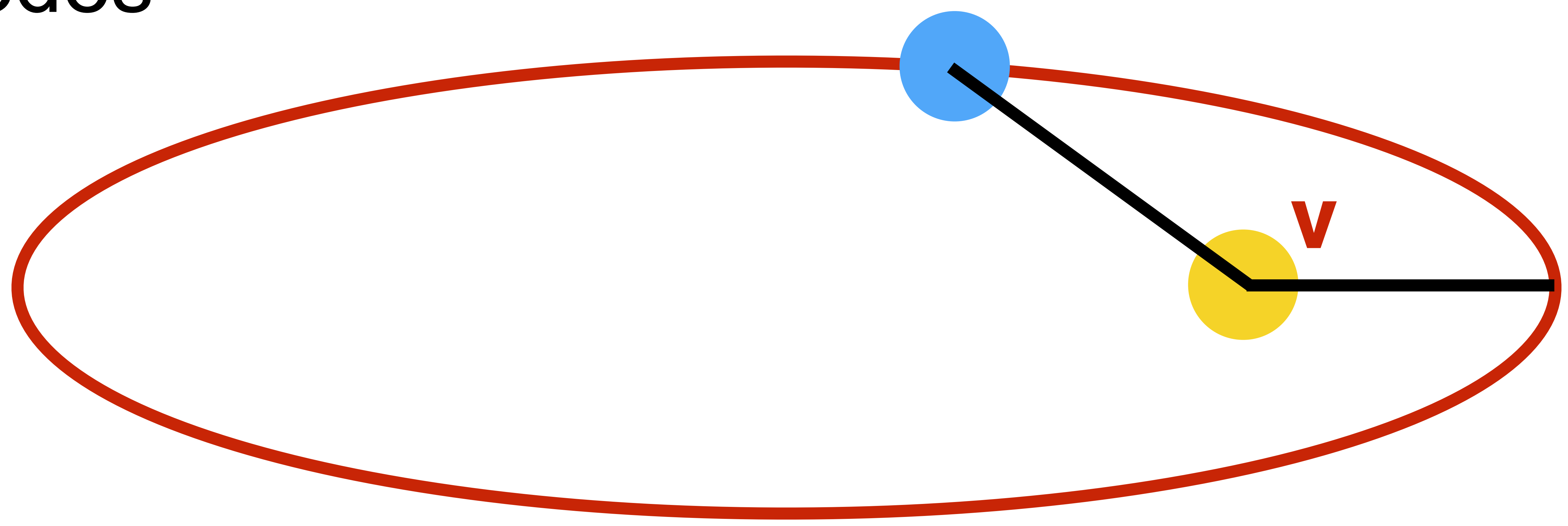
- Masks are encouraged
- No laptops, phones, or other electronic devices during class (I'll let you know in advance if we'll need laptops for an activity) **You may use a tablet to take notes if prefer, but please only use it for note-taking.**
- Remember to bring you response card to class
- Homework 1 will be due tonight at 11:59pm on Canvas

Review of the last class

- If I doubled the distance from the Earth to the Moon, the tidal force felt on Earth would:
  - (A) — get 2 times larger
  - (B) — stay the same
  - (C) — get 2 times smaller
  - (D) — get 4 times smaller
  - (E) — get 8 times smaller

Review of the last class

- What is the angle ν called?
  - (A) — Inclination Angle
  - (B) — Position angle of nodes
  - (C) — True anomaly
  - (D) — Eccentric anomaly
  - (E) — Mean anomaly



Review of the last class

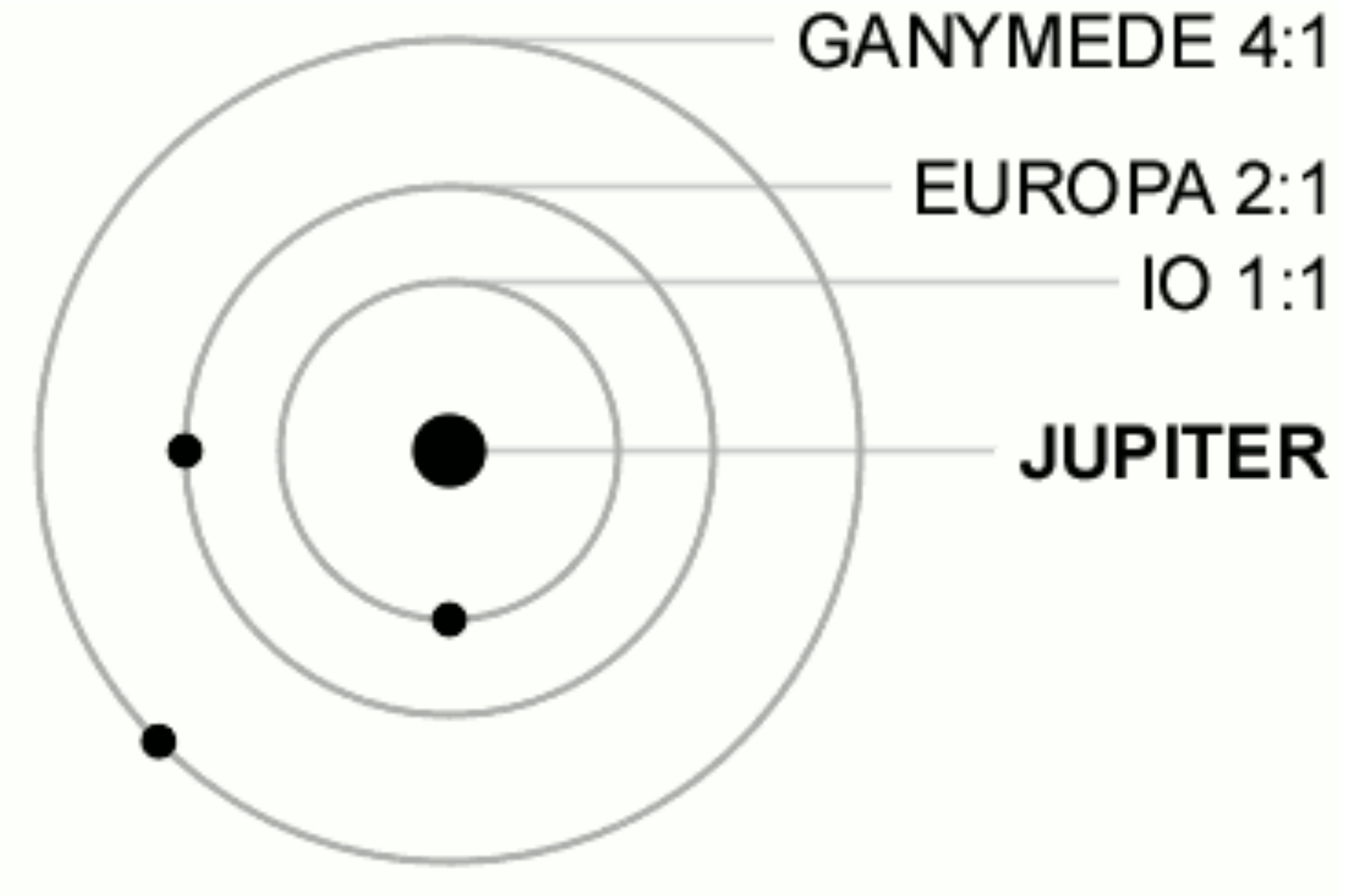
- Due to tidal friction:
  - (A) — Earth days are getting longer, the Moon is moving closer to Earth
  - (B) — Earth days are getting longer, the Moon is moving further away from Earth
  - (C) — Earth days are getting shorter, the Moon is moving closer to Earth
  - (D) — Earth days are getting shorter, the Moon is moving further away from Earth
  - (E) — Earth days are staying the same, the Moon is moving closer to Earth

Review of the last class

- Two stars (A and B), each 1 solar mass, orbit each other. In the reference frame where Star A is not moving, the orbit of Star B has a semi-major axis of 1 AU. What is the orbital period?
  - (A) — 0.5 years (1/2)
  - (B) — 0.71 years (1/1.4)
  - (C) — 1 year
  - (D) — 1.4 years
  - (E) — 2 years

Orbital resonance

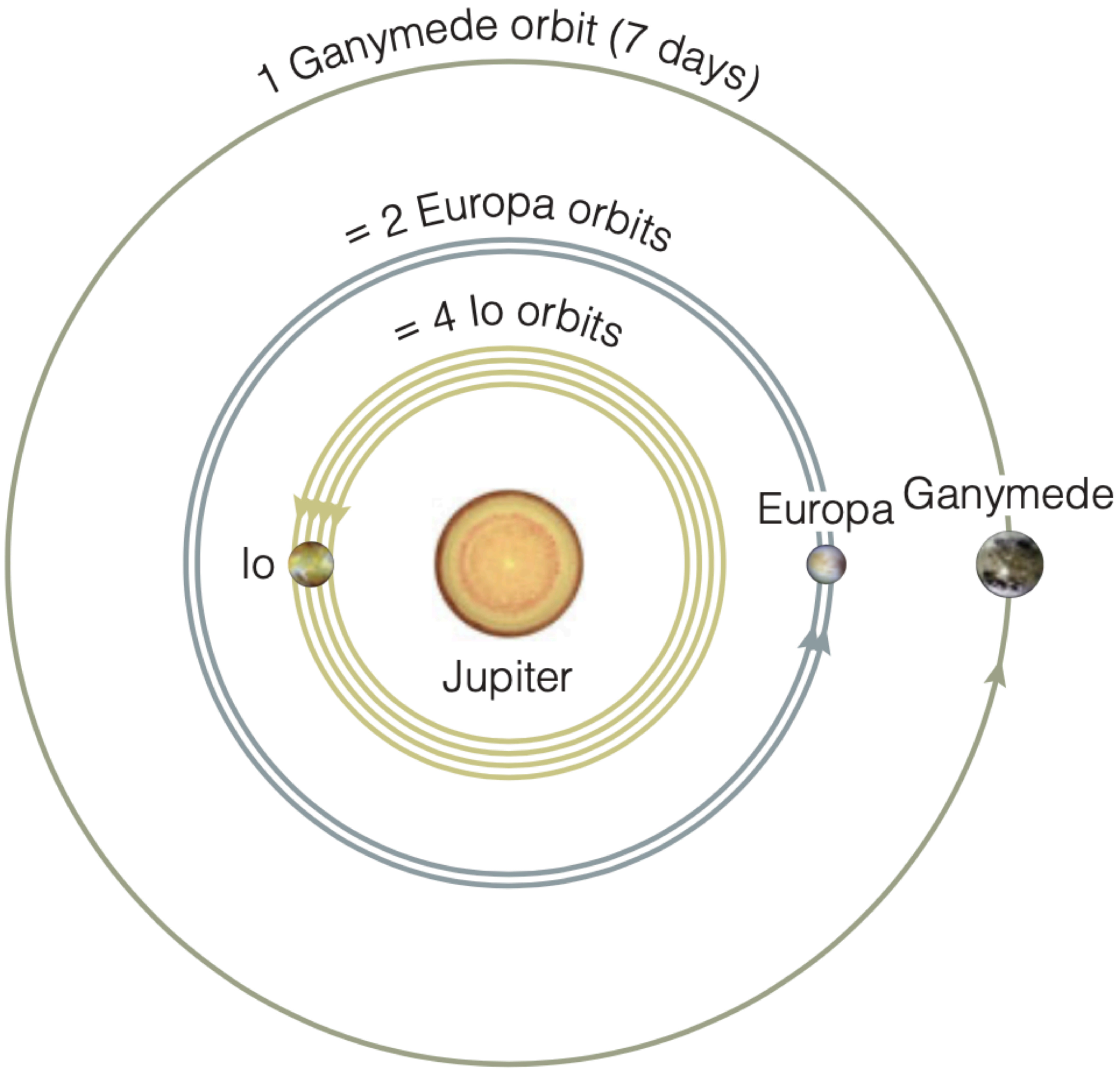
- Moons orbiting a planet (or exoplanets orbiting a star) tend to be in orbital resonance: their orbital periods are very close to integer ratios
- Orbital resonances are more stable, and allow planets/moons to be much more tightly packed than if they weren't in a resonance



wikipedia/WolfmanSF/Matma Rex

Orbital resonance

- Tidal friction caused the Galilean moons to move outward
- Io's orbital period increased until it was in orbital resonance with Europa
- Then both Io's and Europa's orbital period increased (while staying in resonance) until they were in orbital resonance with Ganymede
- These moons are still moving outward, and in a few billion years they may be in orbital resonance with Callisto



"Life in the Universe"

Response Card Question

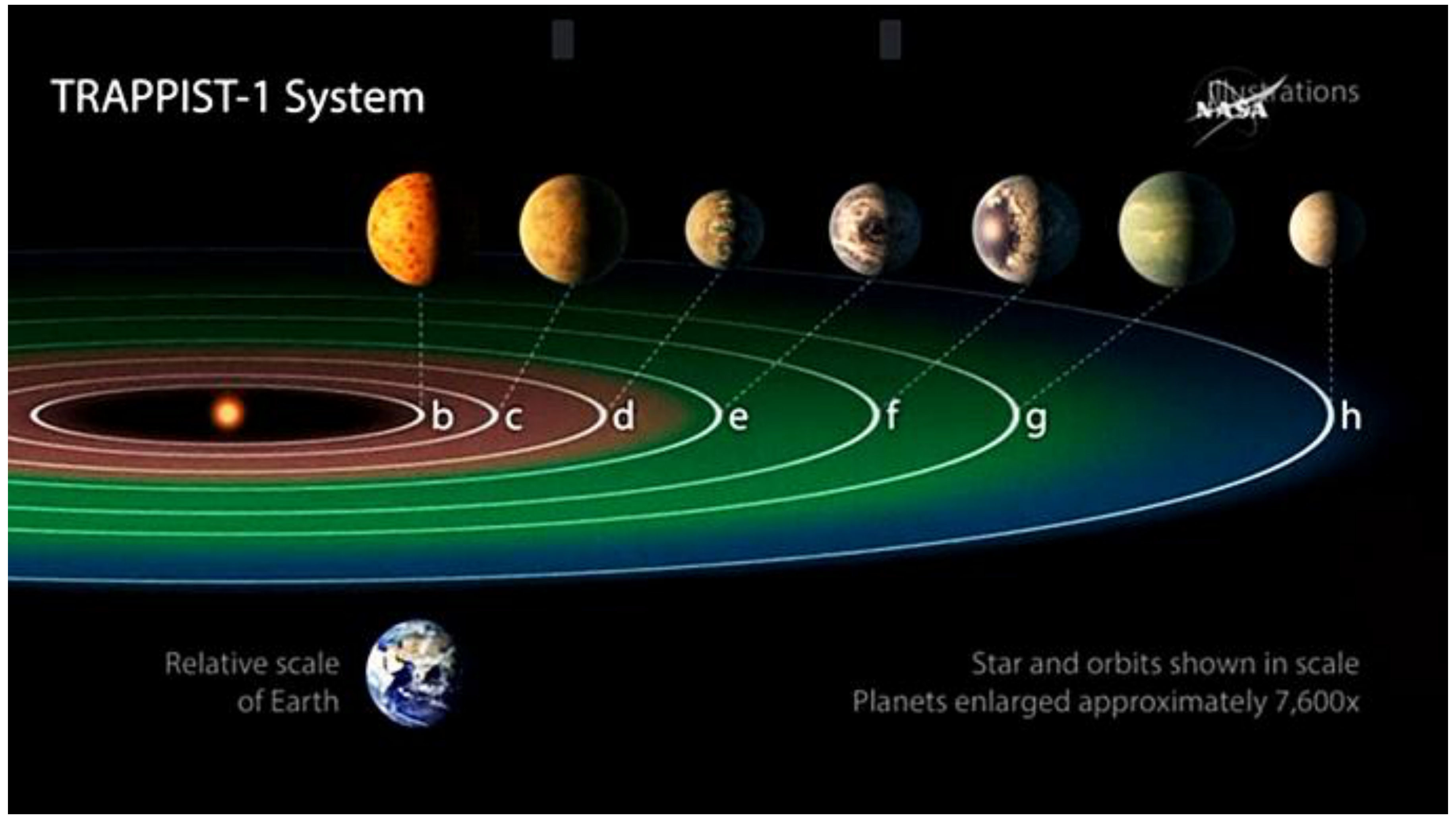
- Consider two moons in a 2:1 resonance, $\dfrac{P_2}{P_1} = 2.$

  The ratio of their semi-major axes, $\dfrac{a_2}{a_1}$ is:

  - (A) — larger than 2
  - (B) — smaller than 2
  - (C) — equal to 2

Orbital resonance

- Exoplanets are also often found in orbital resonance
- The TRAPPIST-1 system has 7 Earth-sized planets with orbital periods all very close to the 3:2 resonance with the next closest planet



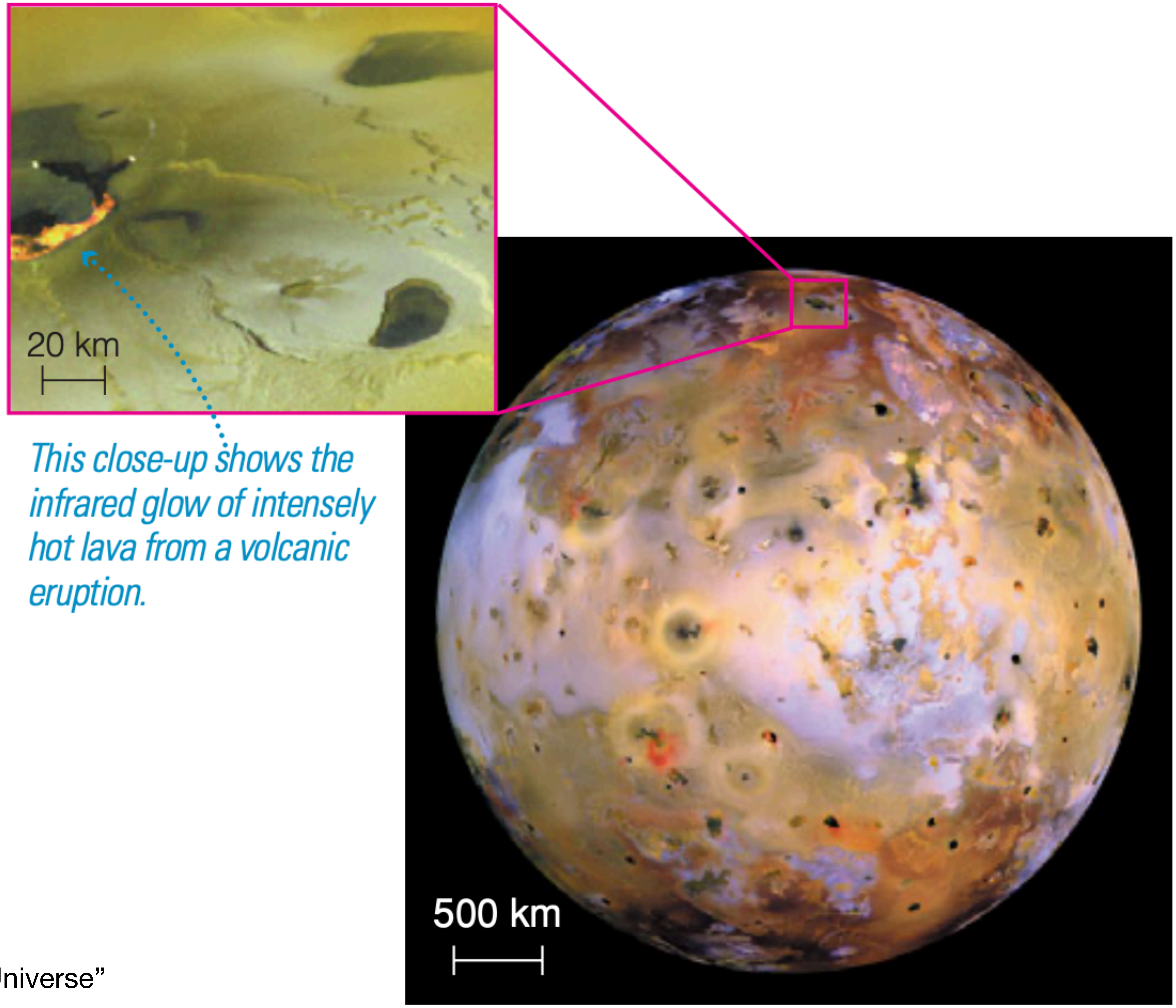
NASA

Planet	Mass	Mean density	Mass	Mean density	Period	Period ratio
–	$m/M_{\oplus}$	$\bar{\rho}/\rho_{\oplus}$	$m/M_{\oplus}$	$\bar{\rho}/\rho_{\oplus}$	P	P_i/P_{i-1}
–	**model A**	**model A**	**model B**	**model B**	–	–
1(b)	0.85	0.66	0.85	0.66	1.510 87081	–
2(c)	1.38	1.17	1.38	1.17	2.421 8233	1.602 932 087
3(d)	0.41	0.89	0.41	0.89	4.049 610	1.672 132 727
4(e)	0.62	0.80	0.62	0.80	6.099 615	1.506 222 821
5(f)	0.68	0.6	0.36	0.32	9.206 690	1.509 388 707
6(g)	1.34	0.94	0.57	0.40	12.352 94	1.341 735 195
7(h)	0.37	1.0	0.086	0.23	18.764	1.518 990 621

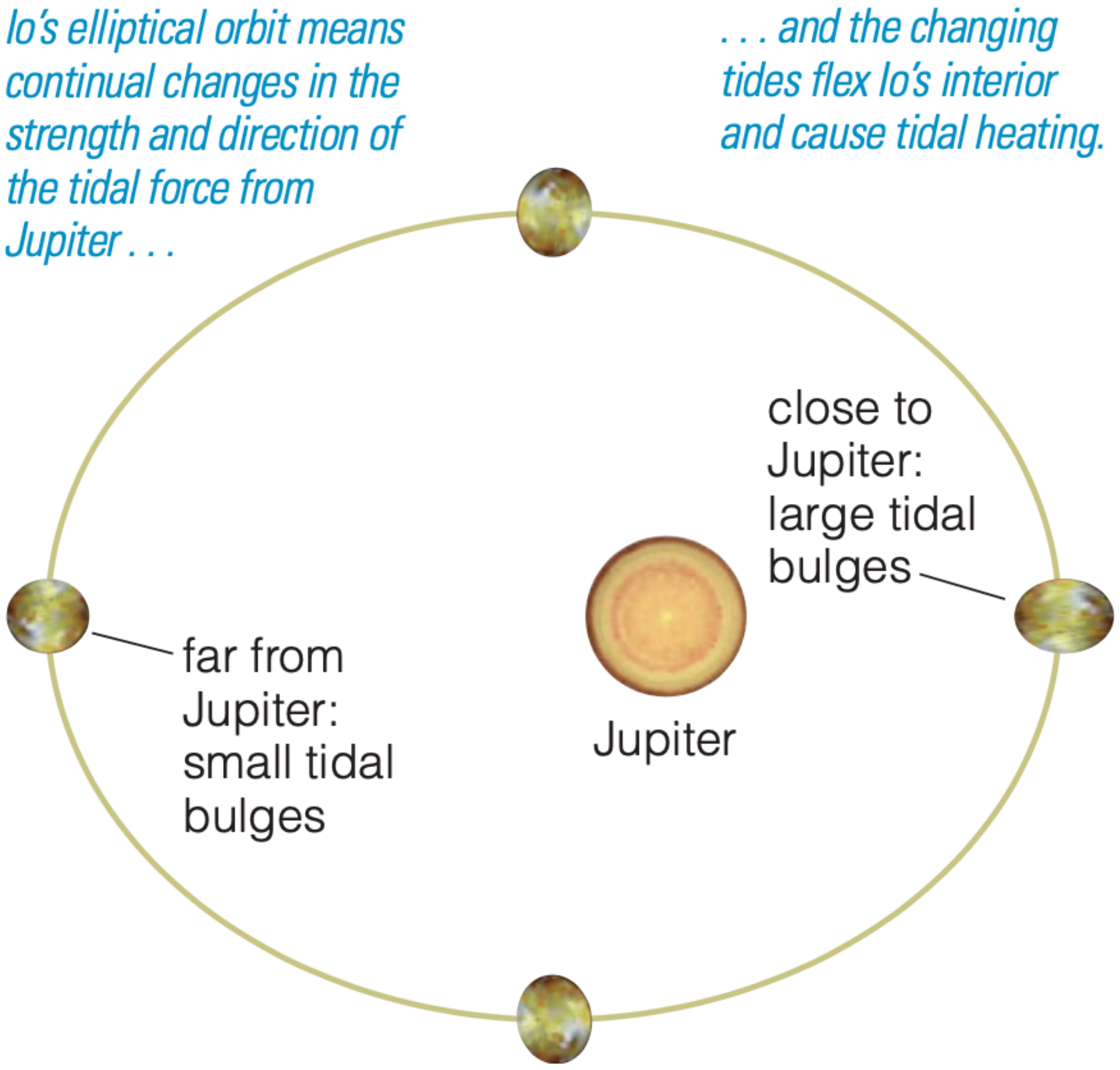
Pabaloizou et al. 2018

Volcanoes on Io

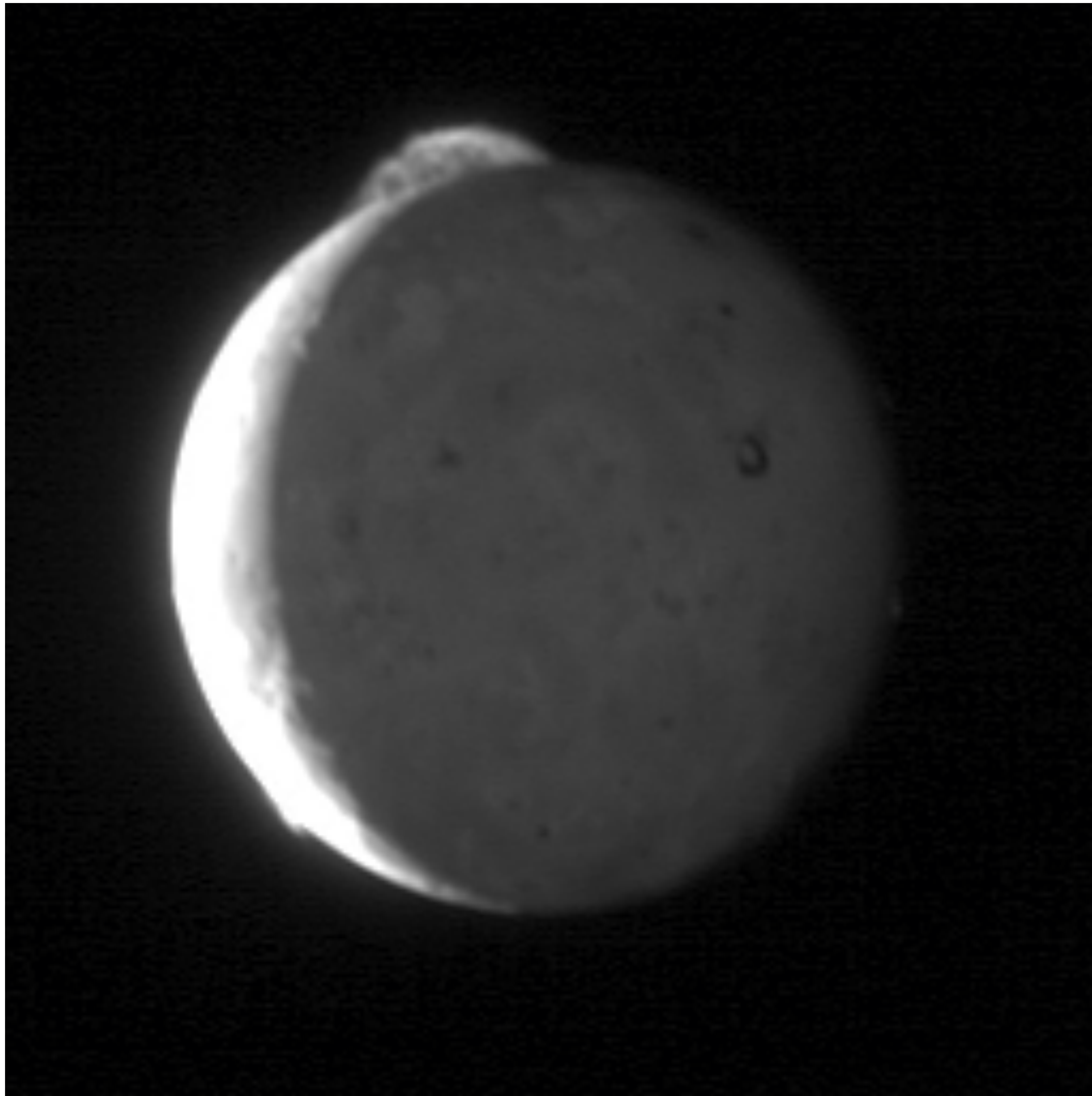
- In synchronous rotation with Jupiter
- Eccentricity is only 0.004 (4 times smaller than Earth's), but enough to cause significant "tidal heating" in Io even today
- Io is the most volcanically active object in our Solar System, with multiple active volcanos



Volcanism from Tidal Heating



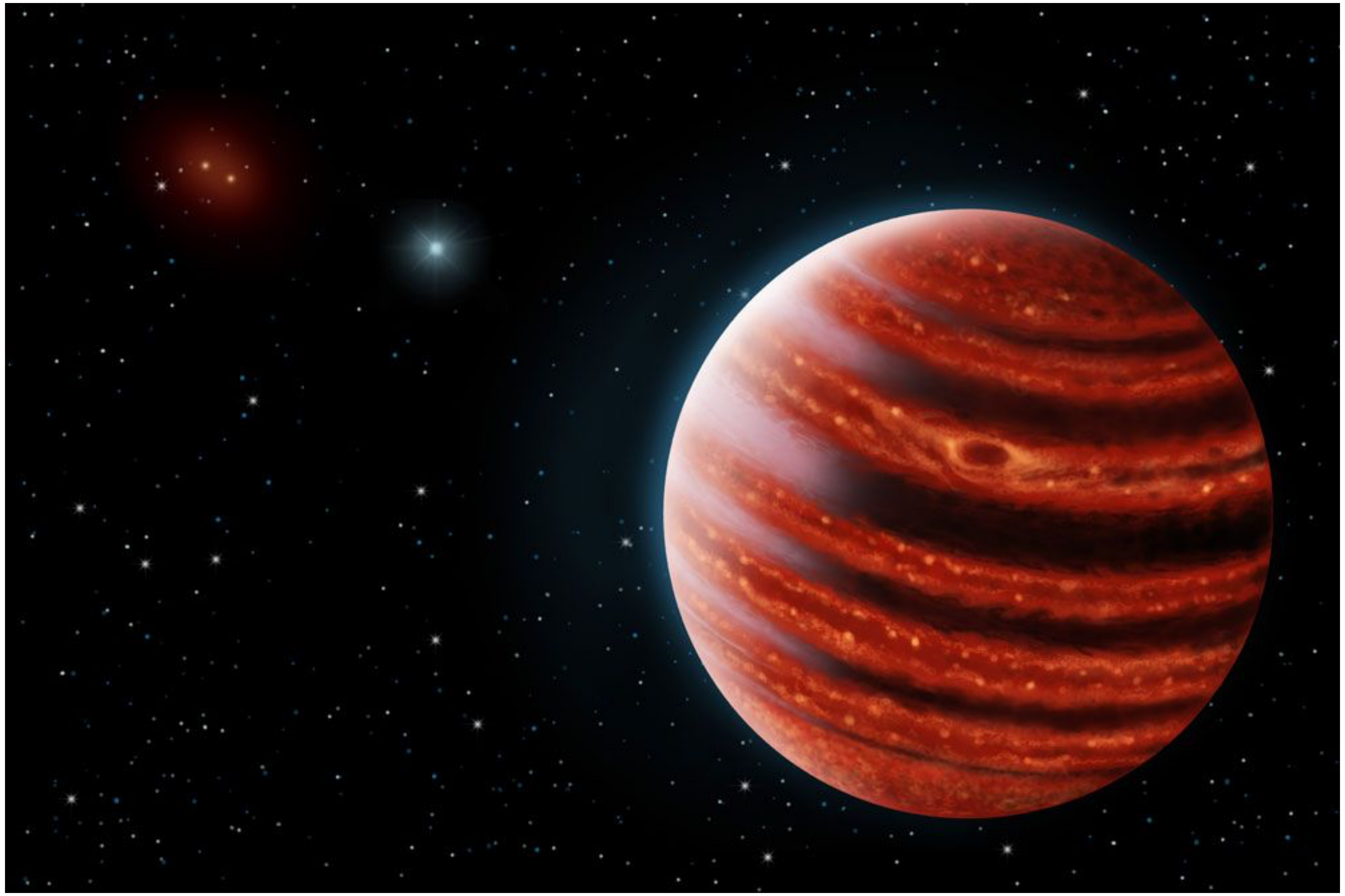
"Life in the Universe"



NASA/New Horizons

The 3-body problem

- There is an exact solution to the 2-body problem (how two objects move as a function of initial conditions and their own gravity)
- There is not an analytic solution to the 3-body problem
- There are approximations in special cases:
  - The restricted 3-body problem (When 1 mass is much smaller than the other 2 masses)
  - Hill's problem (When 1 mass is much larger than the other 2 masses)

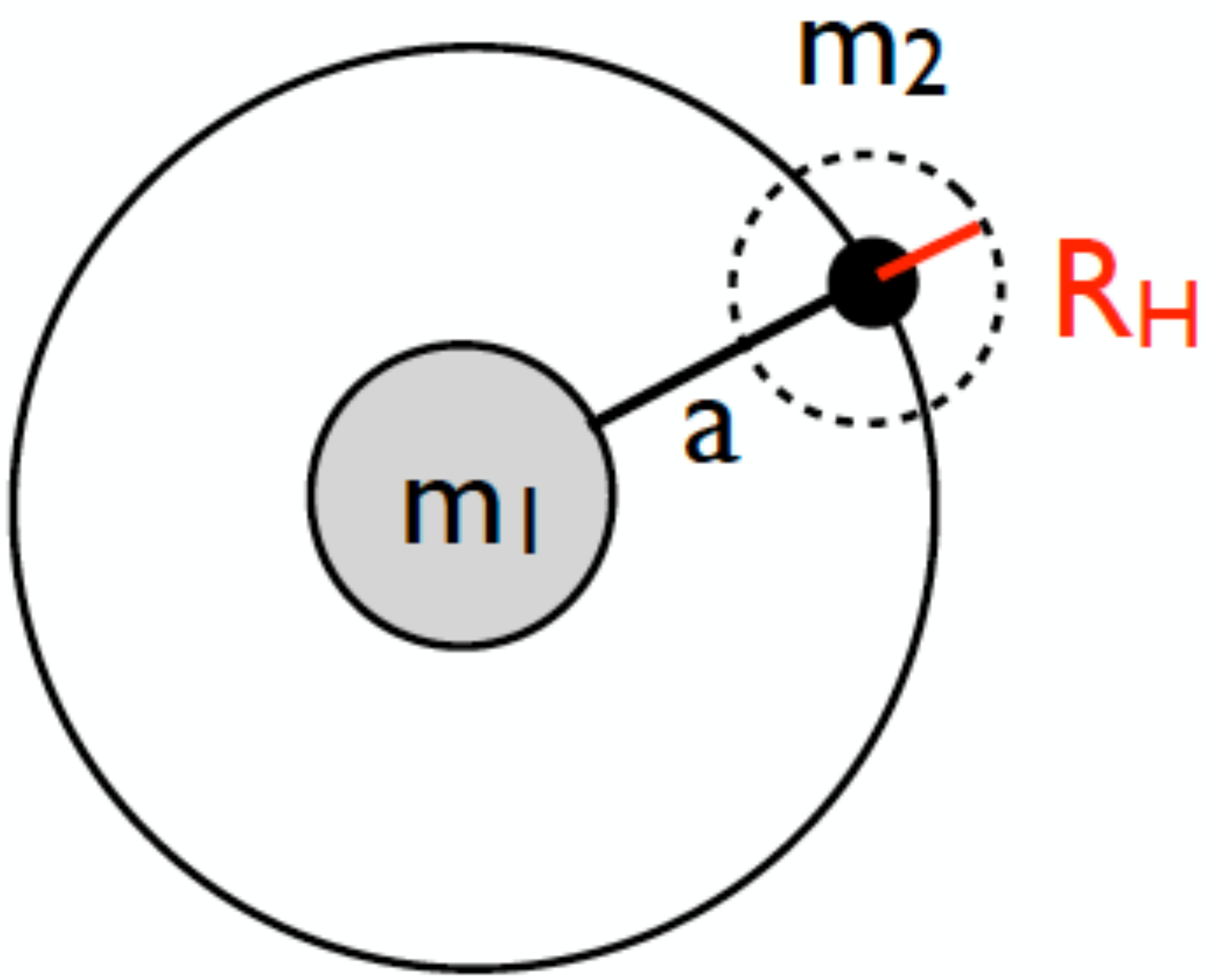


Danielle Futselaar & Franck Marchis

Hill Sphere

- Gravitational sphere of influence around one body (like Jupiter), when there are perturbations from a much more massive body (like the Sun)

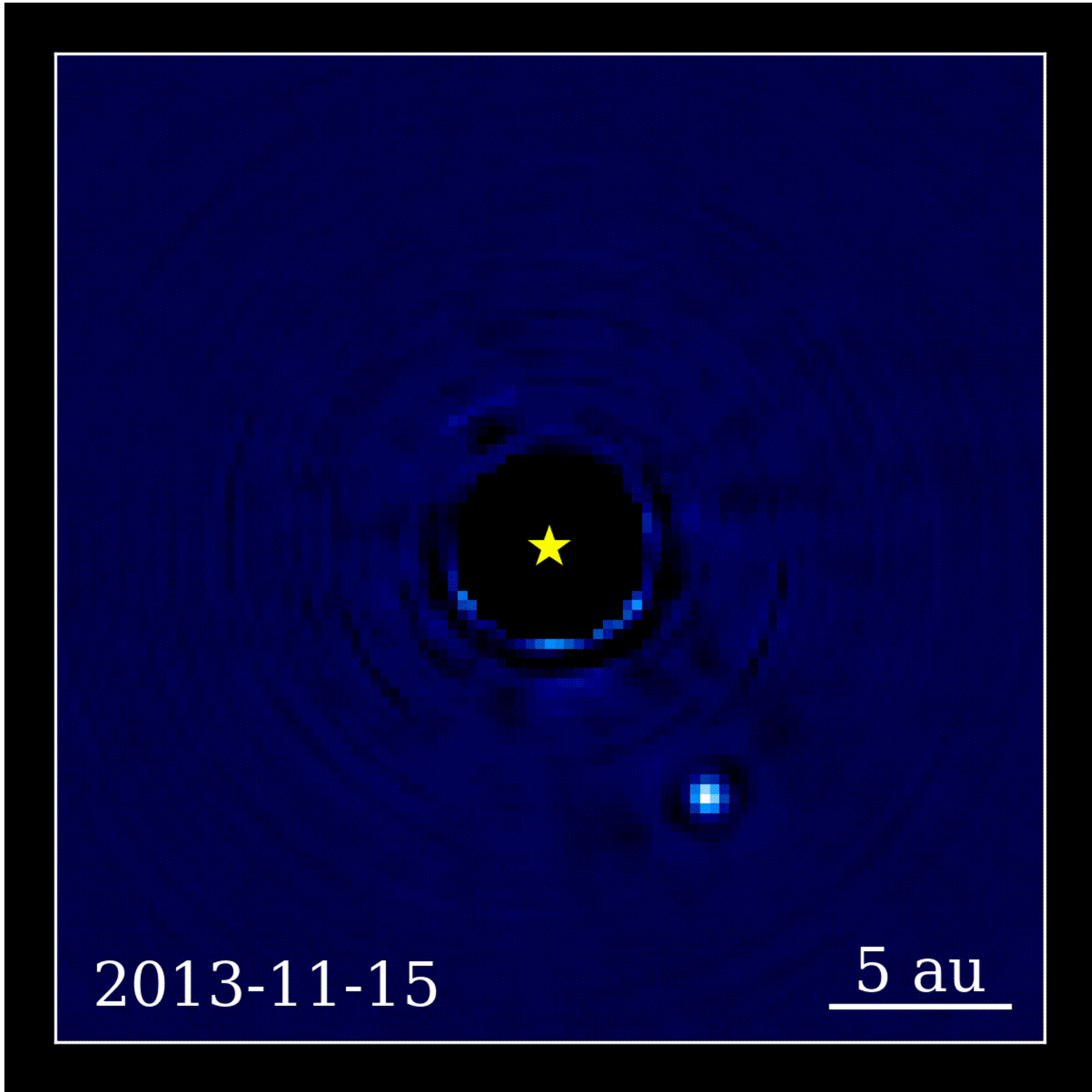
$$R_H = \left(\frac{M_2}{3(M_1 + M_2)} \right)^{\frac{1}{3}} a * (1 - e)$$



- A planet’s moons and rings will be within that planet’s Hill Sphere

beta Pictoris b

- There are 5 main planet-finding techniques: radial velocity, transit, microlensing, direct imaging, and astrometry
- beta Pic b is a directly-imaged planet: 12 times the mass of Jupiter, 9 AU orbit
- beta Pic b was also detected through radial velocity measurements of its host star
- beta Pic b was also detected through the astrometric motion of its host star from the Hipparcos and Gaia satellites
- It has a very edge-on orbit, could it also transit its star (once every 24 years)?

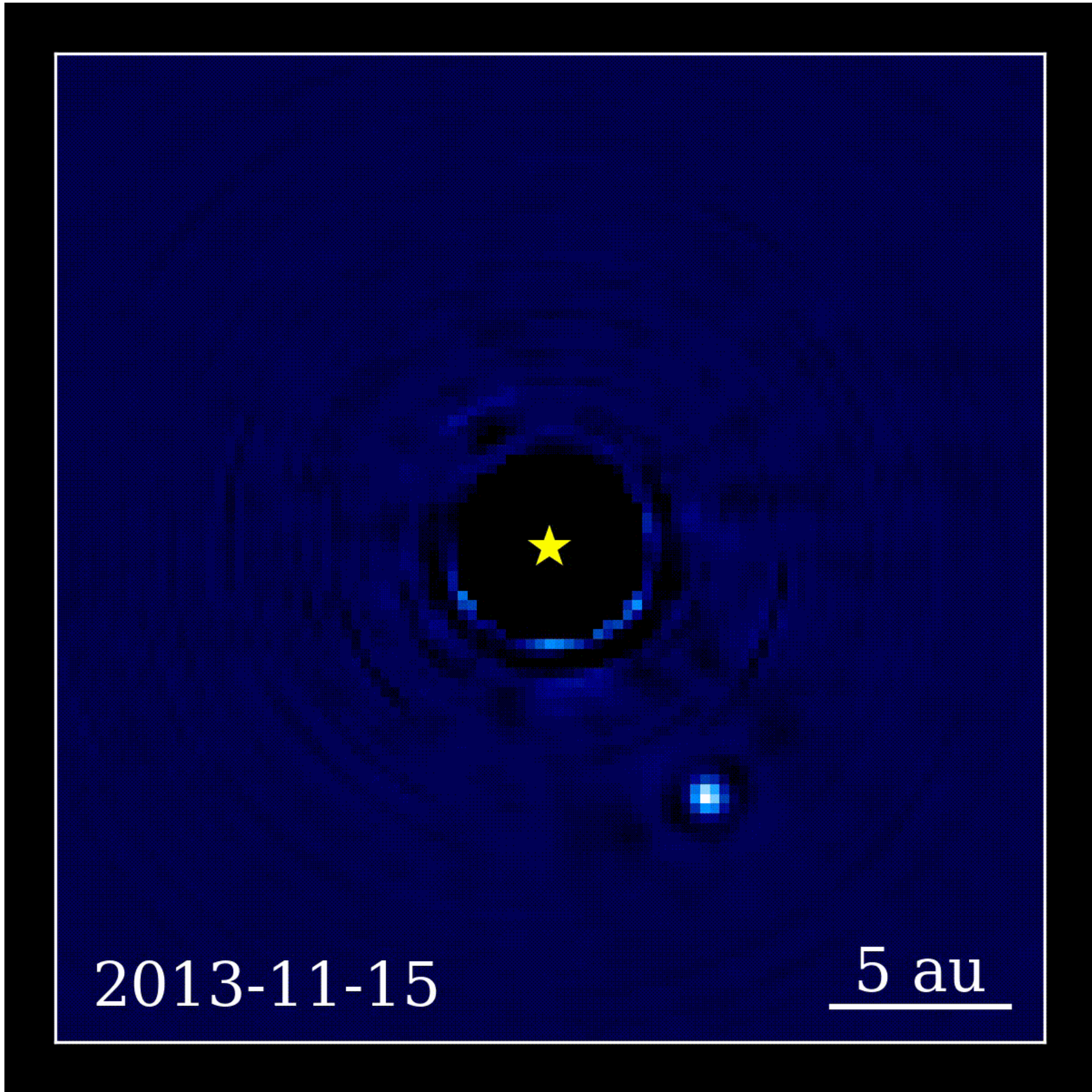


Jason Wang/GPIES

beta Pictoris b

- It has a very edge-on orbit, could it also transit its star (once every 24 years)?
- No

Thus, we concur with Wang et al. (2016) that the astrometry strongly disfavors transit, at the 12.2σ level for Case 3.



Jason Wang/GPIES

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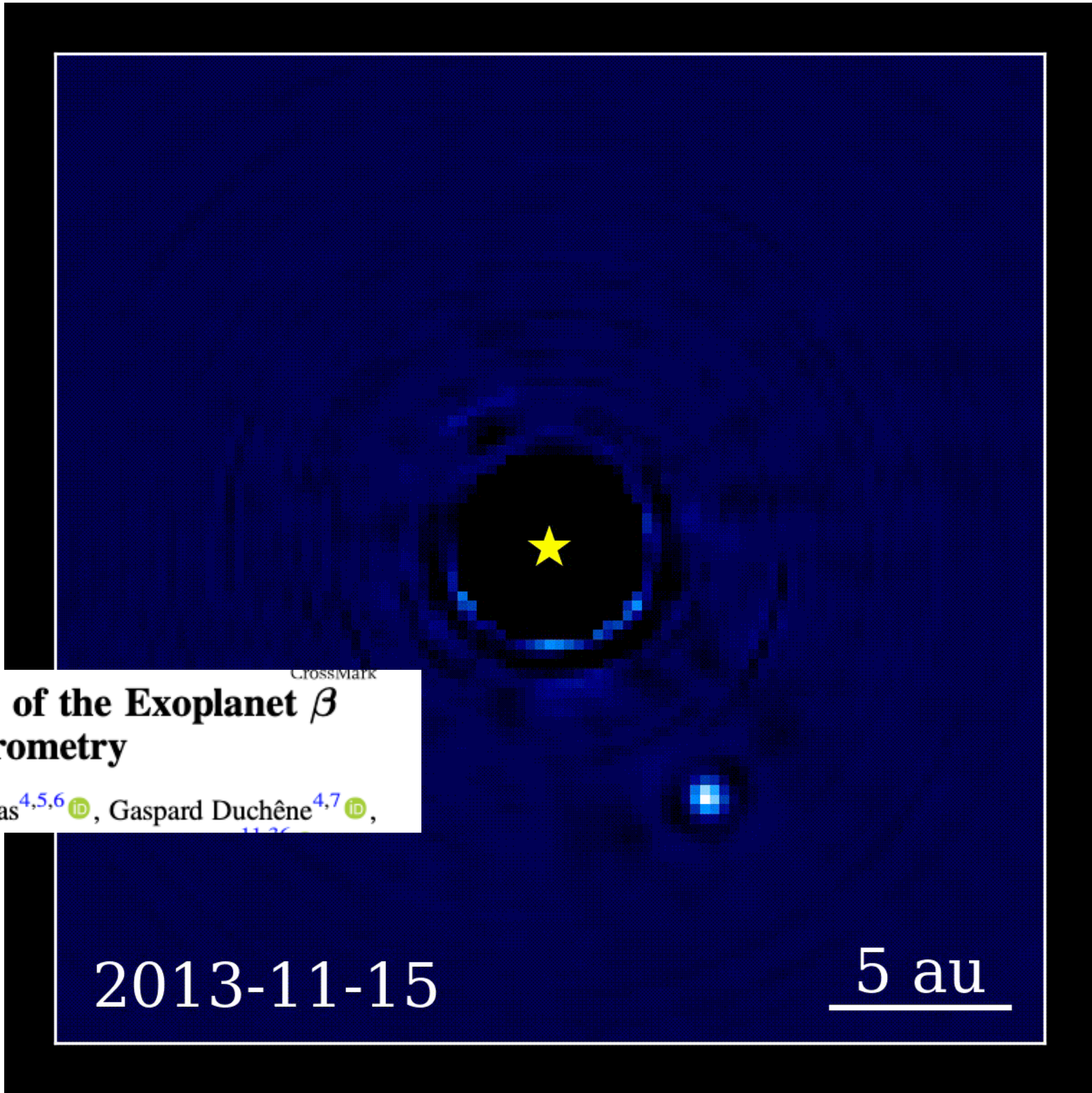
The Gemini Planet Imager Exoplanet Survey: Dynamical Mass of the Exoplanet β Pictoris b from Combined Direct Imaging and Astrometry

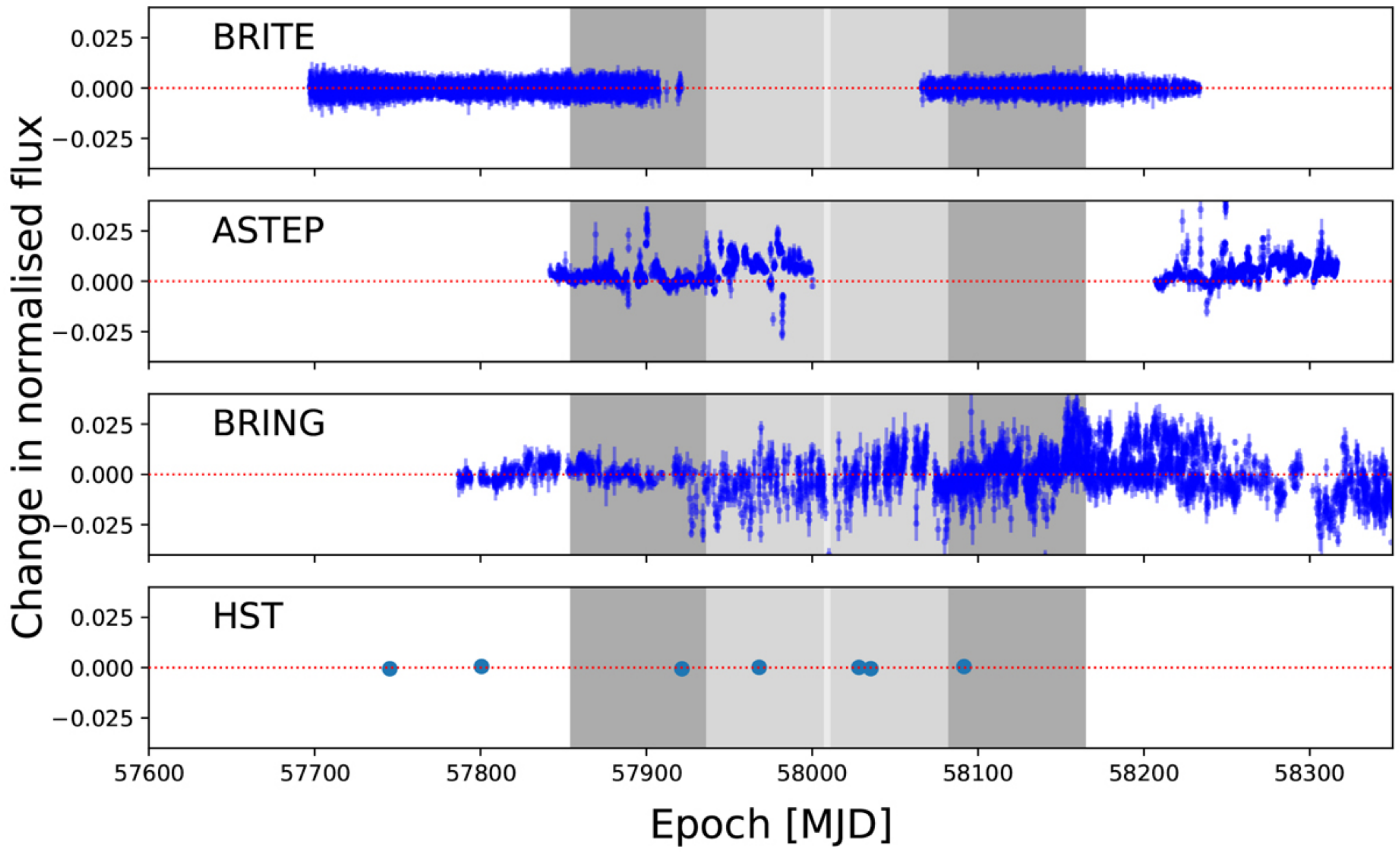
Eric L. Nielsen[1], Robert J. De Rosa[1], Jason J. Wang[2,35], Johannes Sahlmann[3], Paul Kalas[4,5,6], Gaspard Duchêne[4,7],



Jason Wang/GPIES

beta Pictoris b

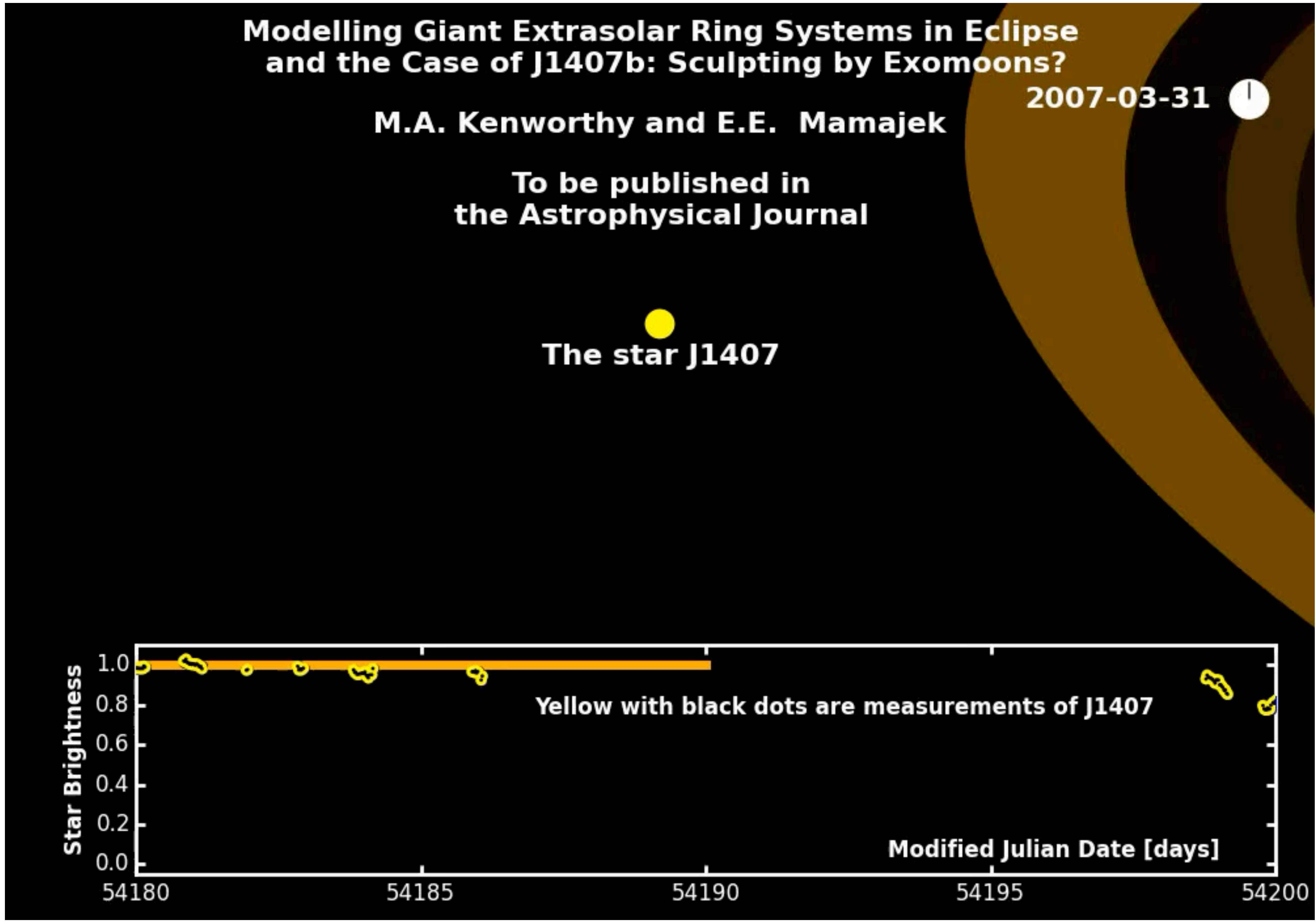
- While the planet did not pass between the star and us, the Hill sphere did in 2017
- Multiple telescope (including cubesats and a specially-built telescope at the South pole) measured the photometry of beta Pic during the Hill sphere transit (shaded regions)
- No detection of moons or rings, and an upper limit on the dust within the Hill sphere



Kenworthy et al. 2021

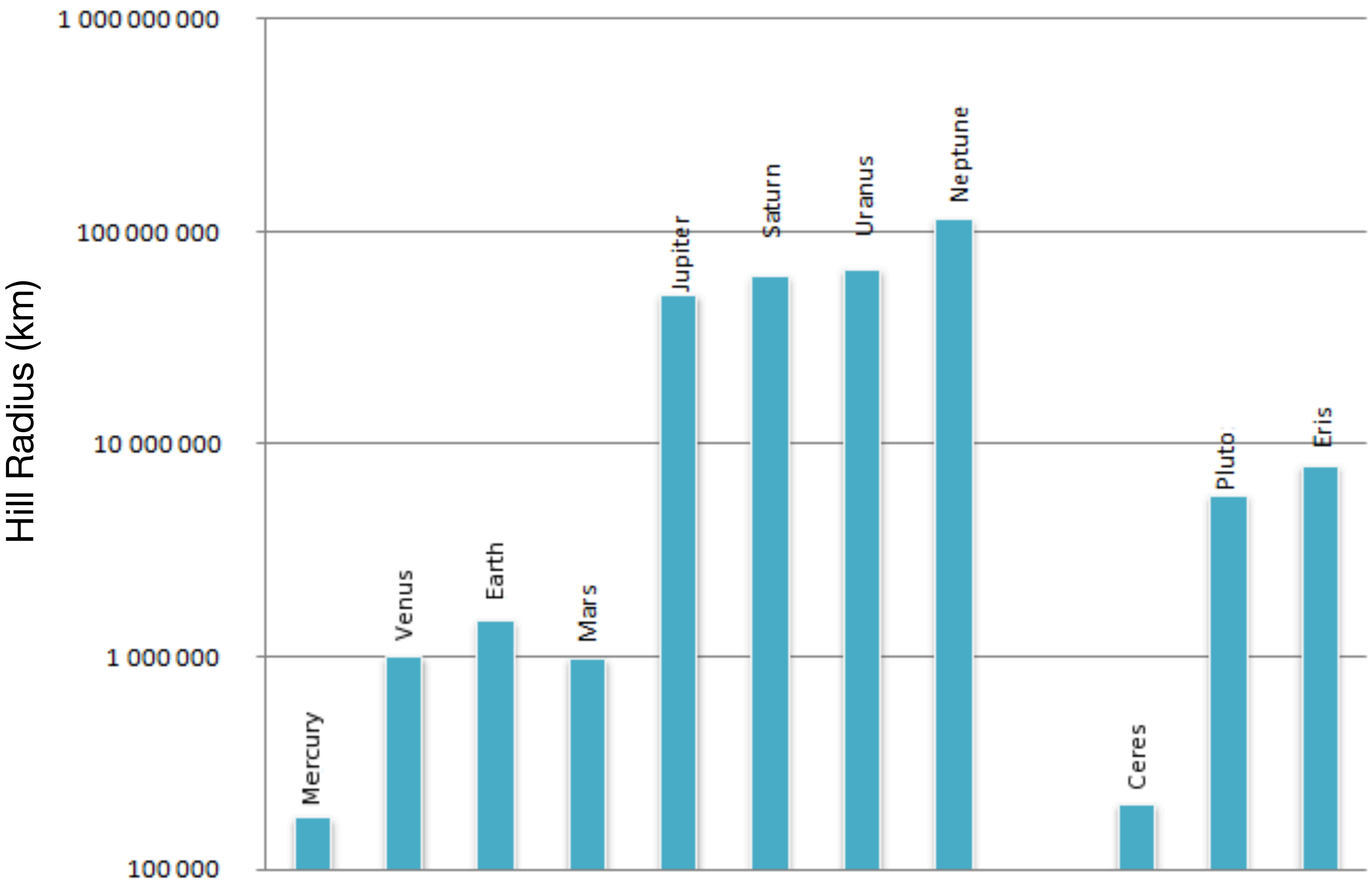
1SWASP J1407

- A SuperWASP light curve showed quasi-periodic dimming of the star 1SWASP J140747.93-394542.6
- Consistent with a planet (which does not transit) and a very large ring system (which does transit)
- The upper limit for the size of a ring system will be a planet's Hill Sphere



Kenworthy and Mamajek 2015

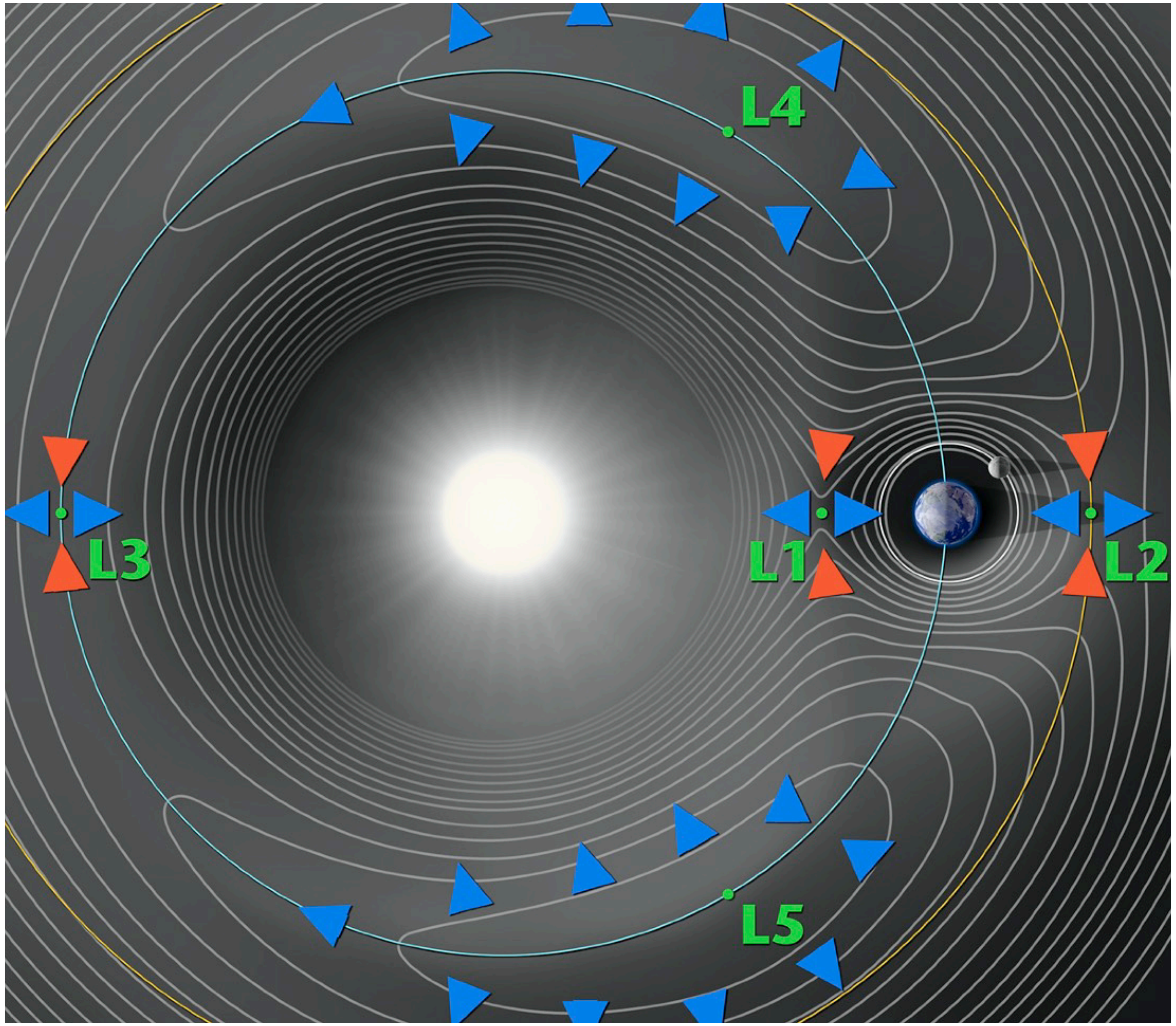
Hill Spheres



wikipedia

Lagrange Points

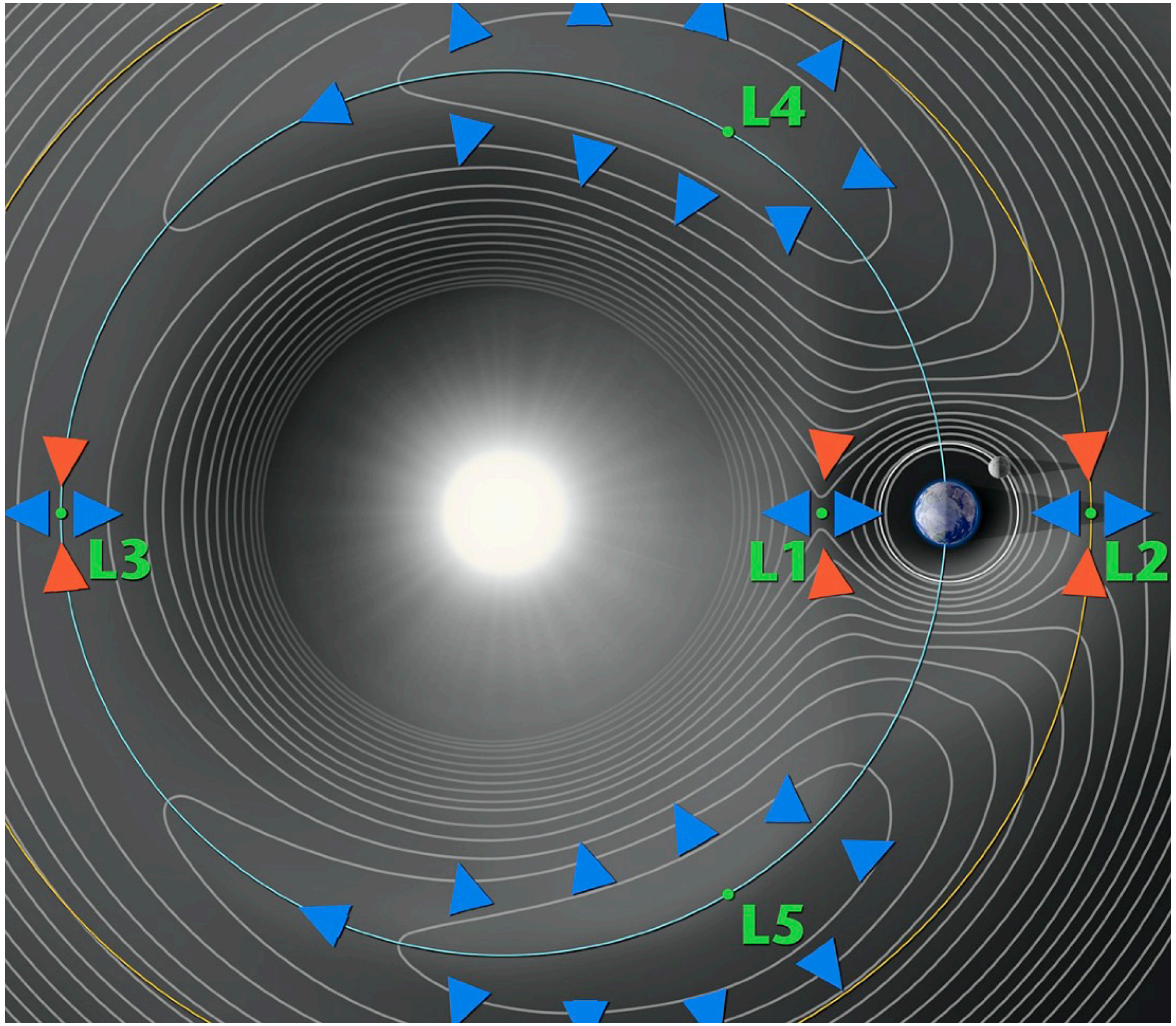
- In a system with two large masses (say, the Earth and Sun), in a reference frame rotating with orbital motion:
  - There are 5 points where centripetal acceleration and gravitational forces from the two objects means a test particle will have an orbital period equal to that of the two objects
- Test particles (masses much smaller than the two large objects) will feel no net force in this rotating frame



solarsystem.nasa.gov

Lagrange Points

- L1, L2, and L3 are unstable over ~23 days
- L4, L5 are stable, if $\frac{M_1}{M_2} \gtrsim 25$
- James Webb Space Telescope is in the Earth/Sun L2 point, so the sunshield can block both while the antenna points to Earth
  - Thruster firings are done every 3 weeks to keep Webb at L2



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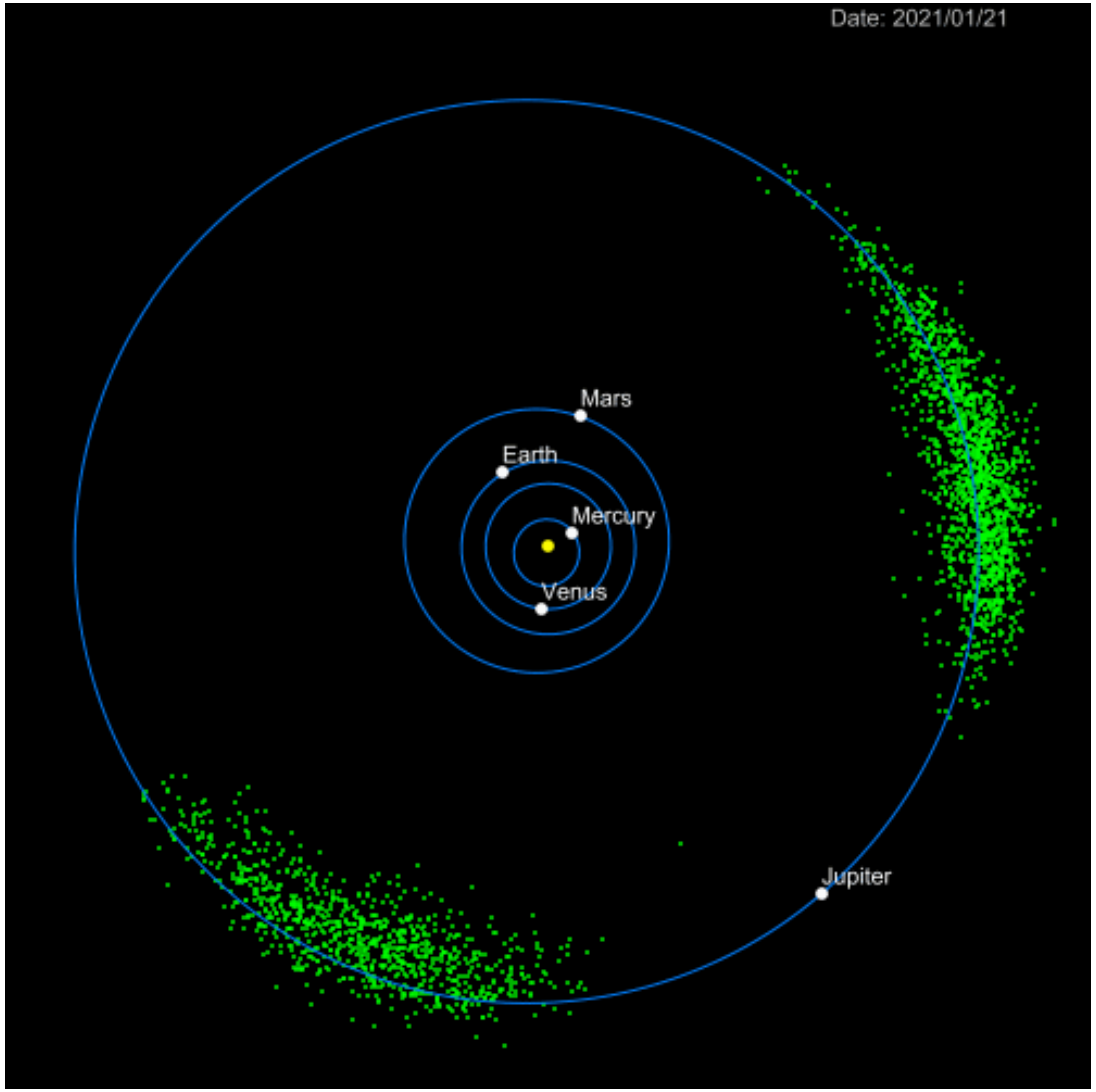
Trojans

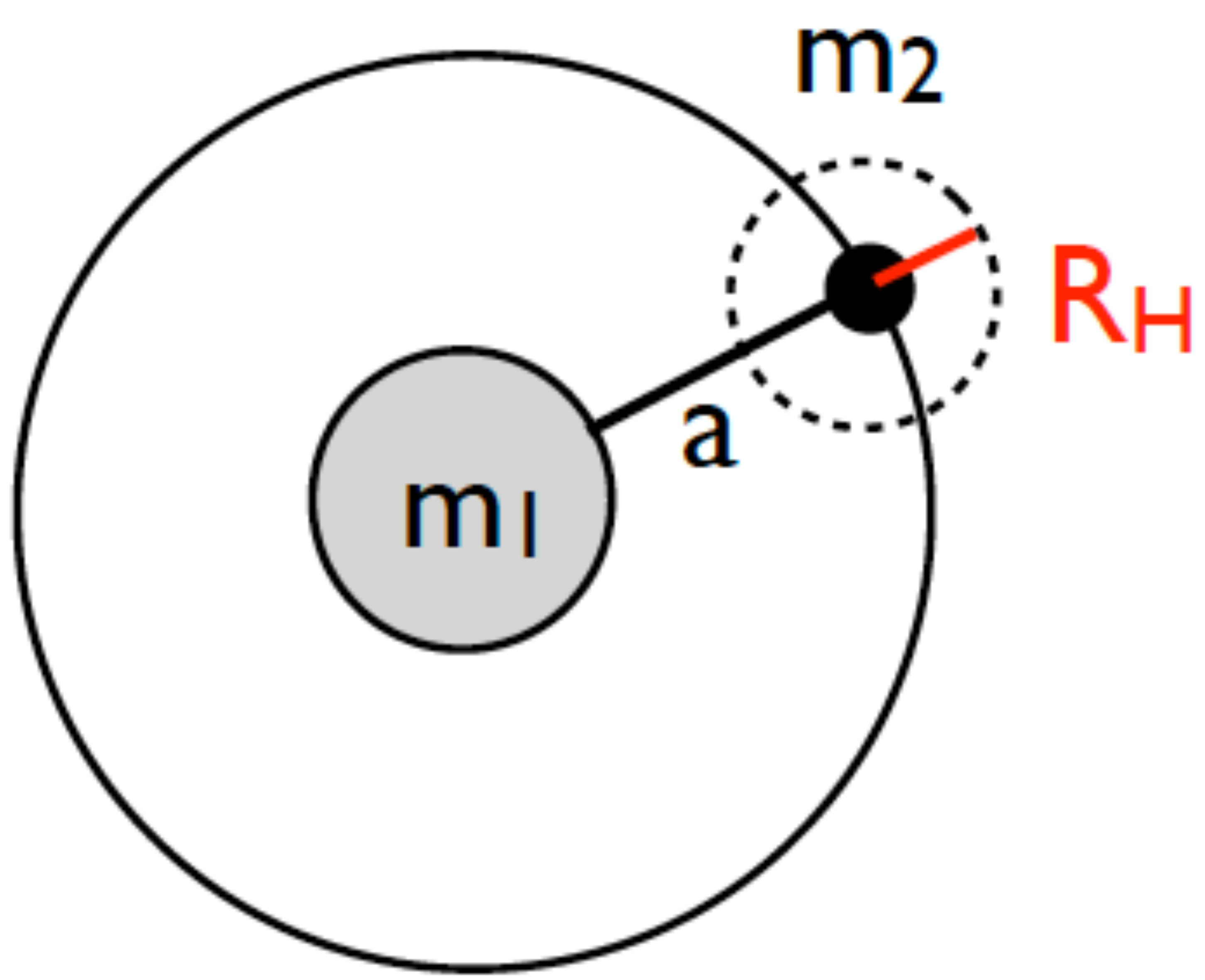
- Trojan asteroids are orbit in the Jupiter/Sun L4 and L5 Lagrange points
- They have distinct properties compared to main belt asteroids



NASA/Petr Scheirich

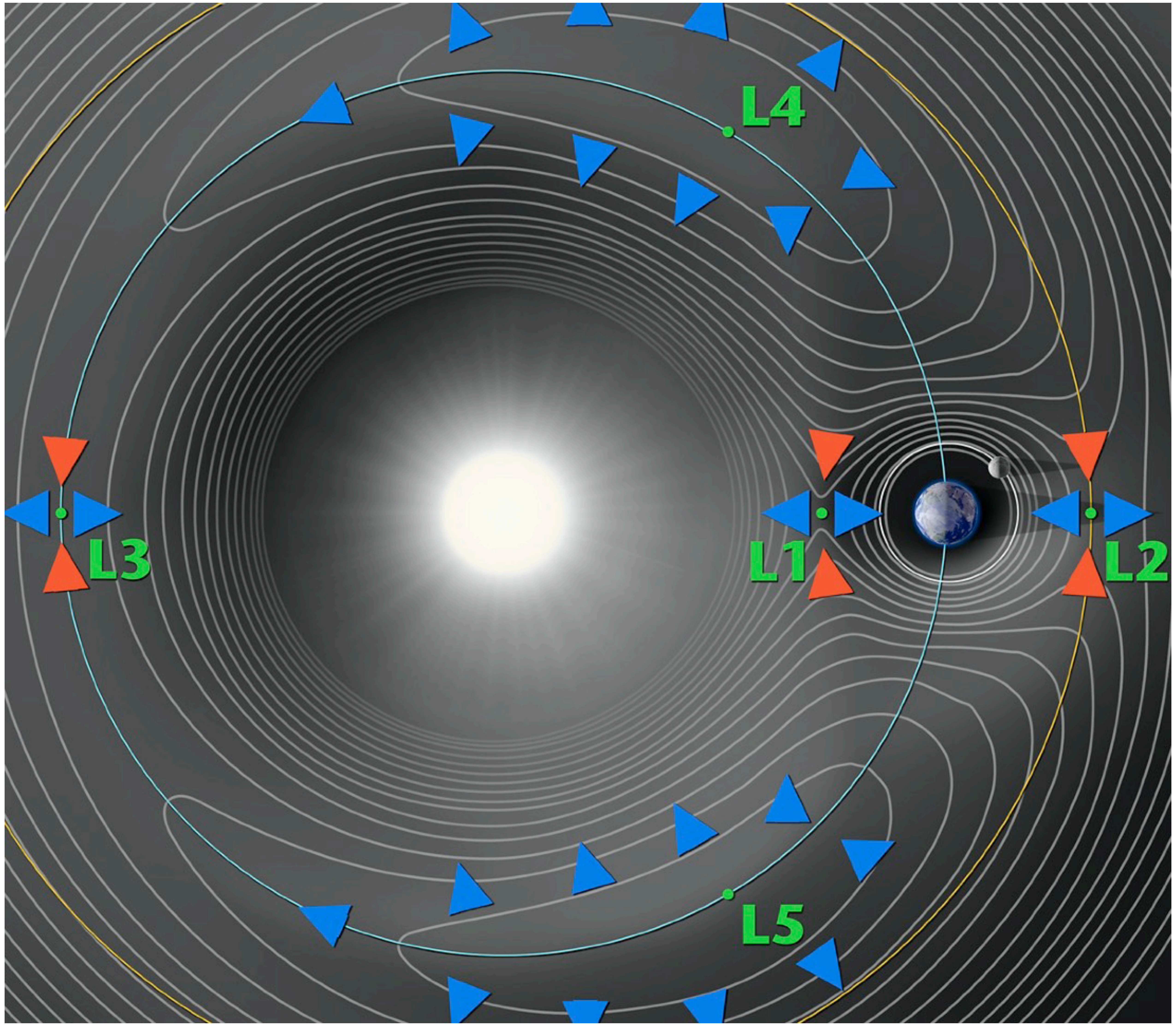
Hill Sphere

- In the Sun-Jupiter system, for example, Hill sphere is derived by considering 3 vector fields:
    - gravity due to the Sun
    - gravity due to Jupiter
    - centrifugal force in reference frame rotating around Sun with same angular frequency as Jupiter
- Hill sphere is the largest radius within which the sum of these 3 fields is directed toward Jupiter



Lagrange Points

- Hill Sphere extends between L1 and L2 points
- Beyond the Hill sphere, if an object tries to orbit the planet, tidal forces from the Sun would perturb orbit, and the object would eventually orbit the Sun



solarsystem.nasa.gov

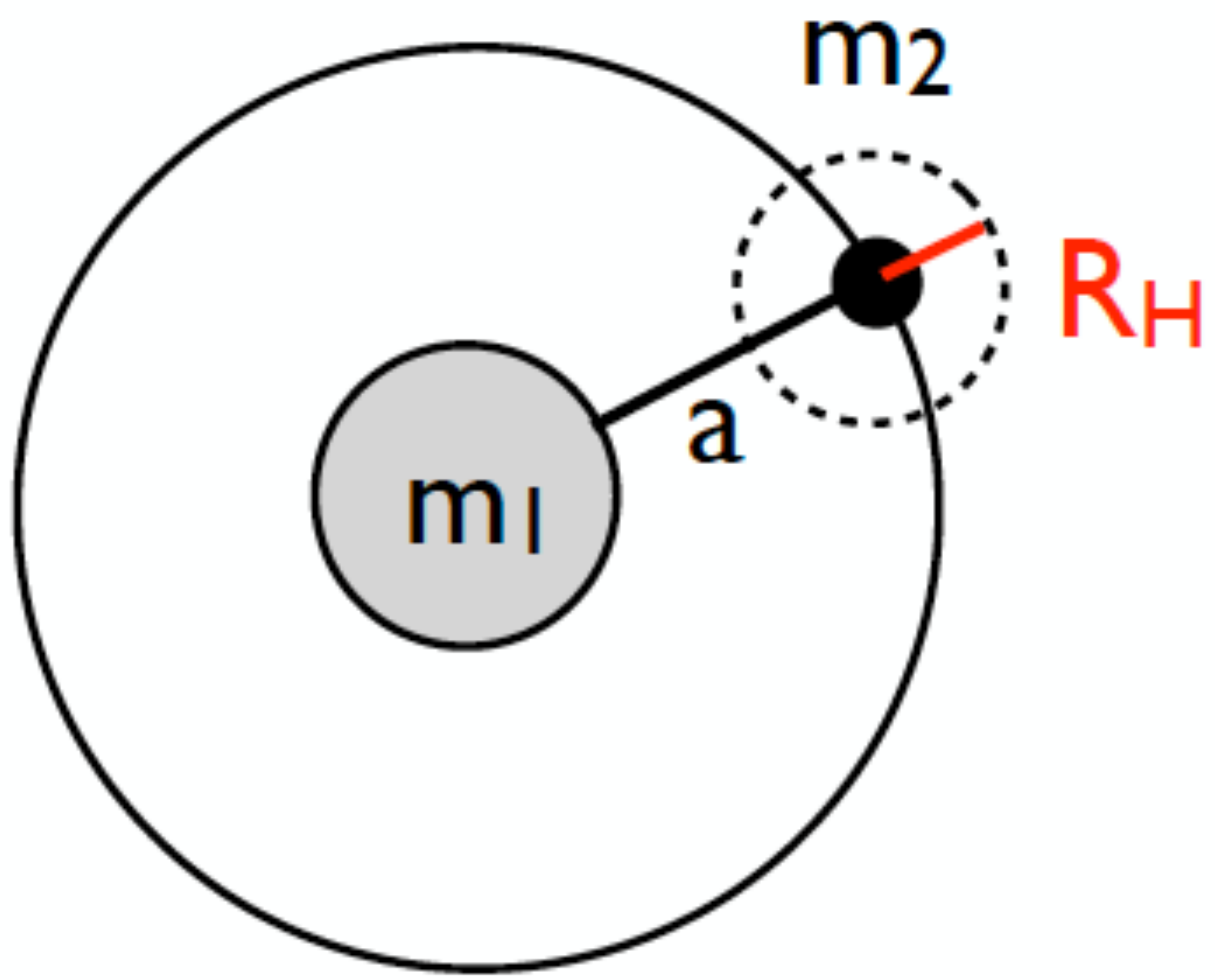
Break

Roche Limit

- Hill Sphere sets maximum size of a moon's orbit, Roche limit sets the minimum size:
    - The tidal force gets larger as distance gets smaller:

$$F_{tidal} = \frac{2GMmr}{d^3}$$

- As distance gets smaller, tidal bulges get larger and larger
- Too close, and the moon's cohesion forces can't counteract tidal forces: disruption

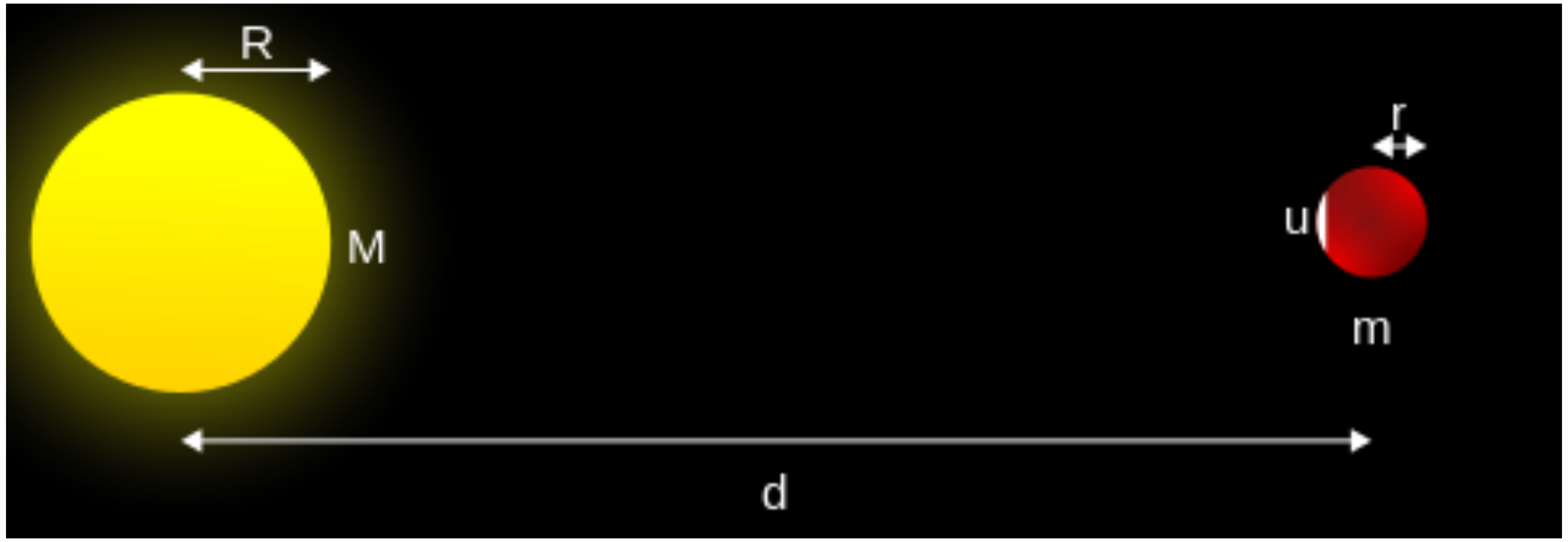


Roche Limit

- Roche limit is where tidal force on the near edge of a moon, u, exactly balances the force of gravity between u and the rest of the moon:

- $$F_{tidal} = \frac{2GMur}{d^3}$$

- $$F_g = \frac{Gmu}{r^2}$$



- Setting these equal:

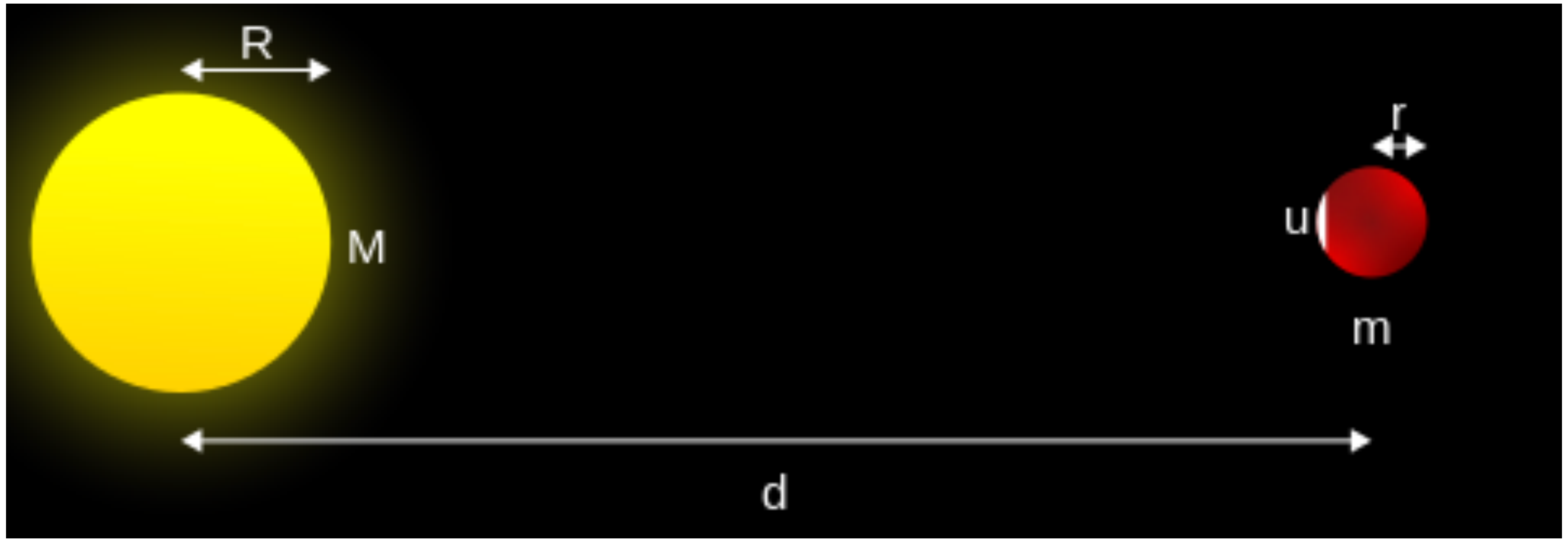
$$\frac{2GMur}{d^3} = \frac{Gmu}{r^2}$$

Roche Limit

- $$\frac{2GMur}{d^3} = \frac{Gmu}{r^2}$$

- Solving for d, we get the Roche Limit:

$$d = r\left(\frac{2M}{m}\right)^{\frac{1}{3}}$$

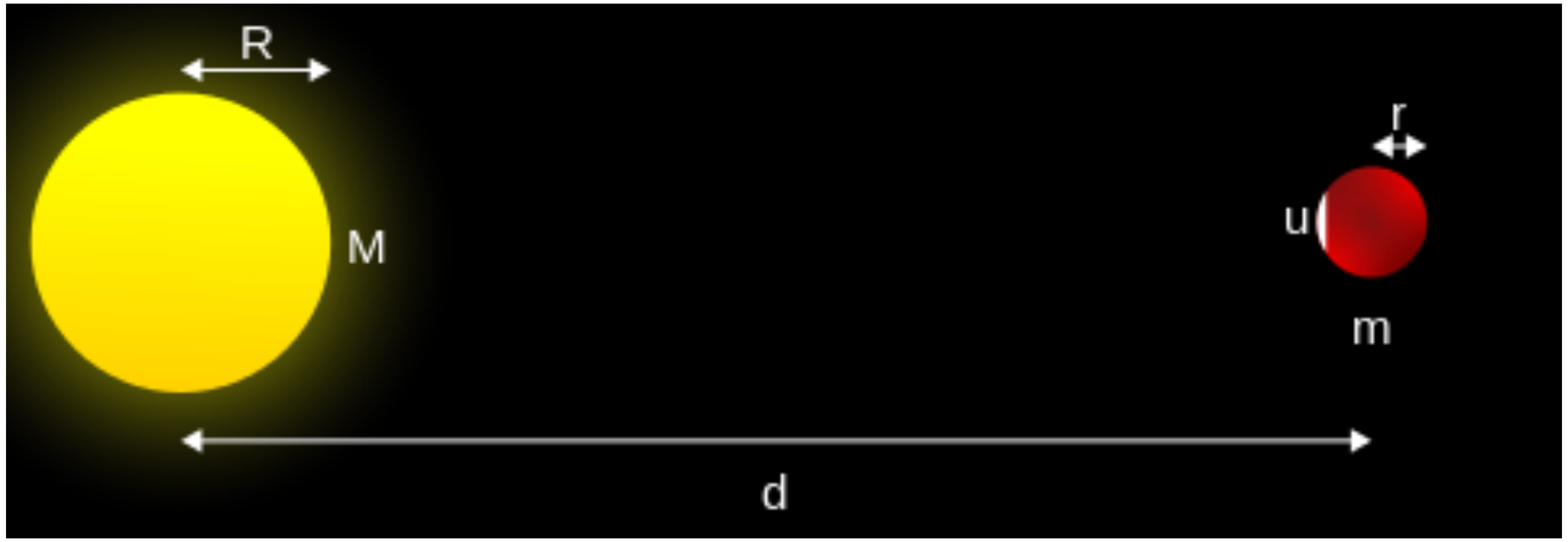


Roche Limit

- $d = r\left(\frac{2M}{m}\right)^{\frac{1}{3}}$

- Can also express this in terms of density of the two objects:

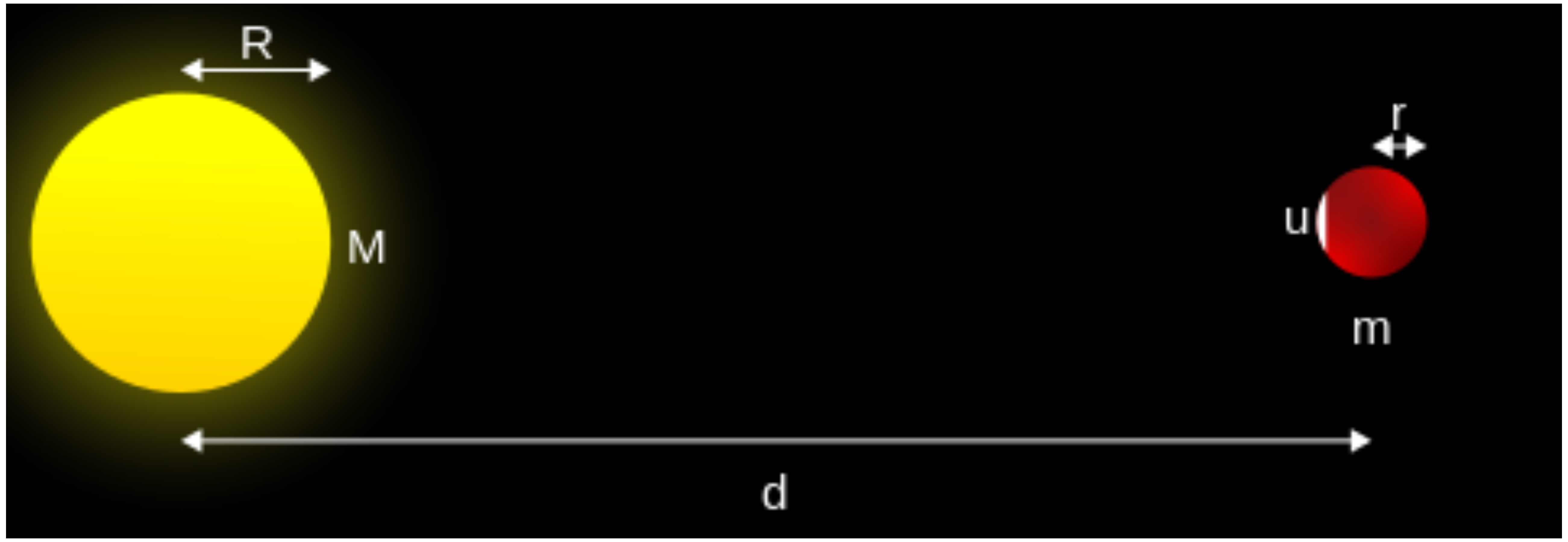
$$M = \frac{4}{3}\pi R^3 \rho_M \qquad m = \frac{4}{3}\pi r^3 \rho_m$$



$$d = r\left(\frac{2\rho_M R^3}{\rho_m r^3}\right)^{\frac{1}{3}} = R\left(\frac{2\rho_M}{\rho_m}\right)^{\frac{1}{3}} \approx 1.26R\left(\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}}$$

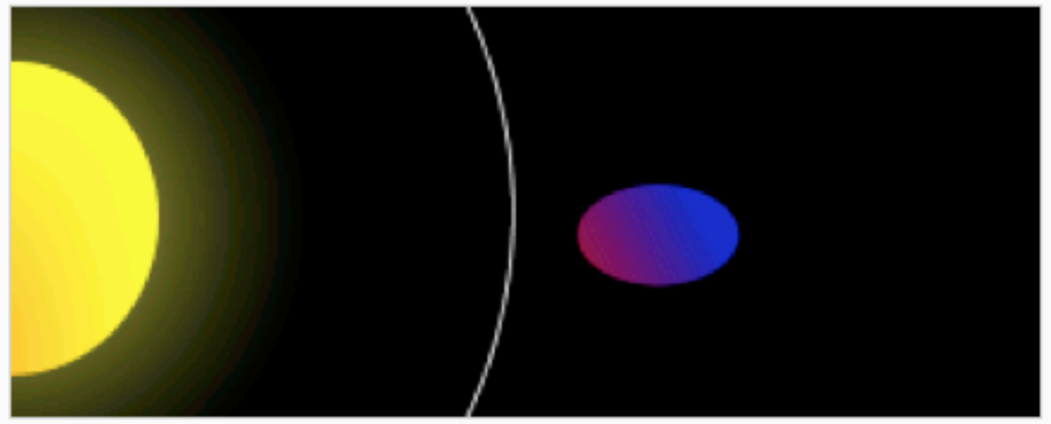
Roche Limit

- $d \approx 1.26R\left(\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}}$
- That’s for a rigid body. If the surface is a fluid:
- $d \approx 2.44R\left(\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}}$

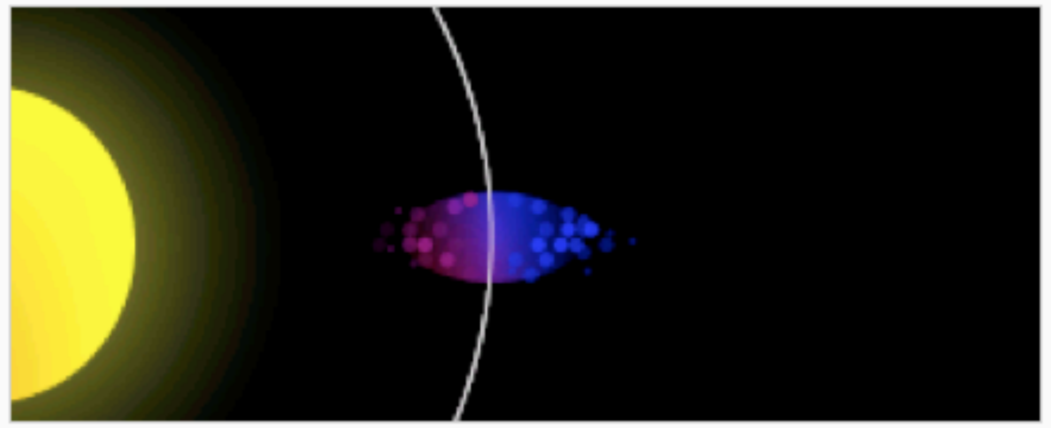


Roche Limit

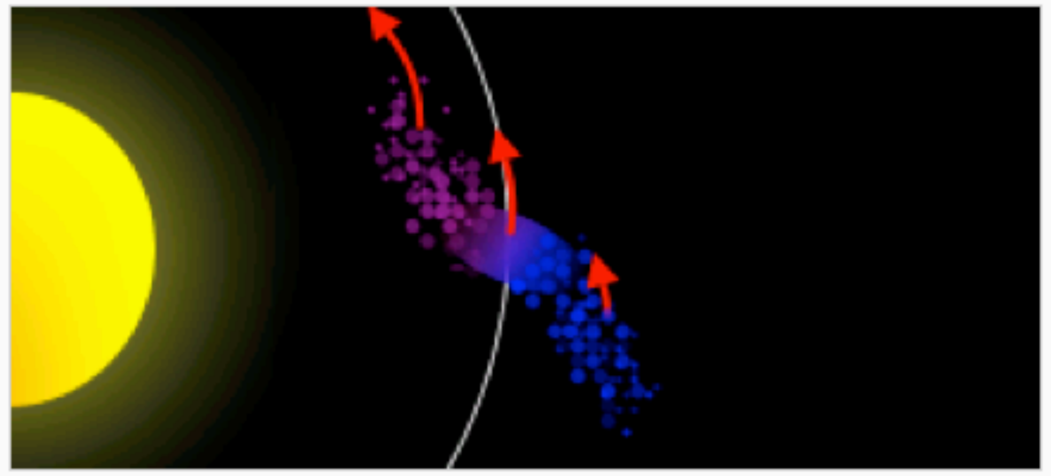
Consider an orbiting mass of fluid held together by gravity, here viewed from above the orbital plane. Far from the Roche limit the mass is practically spherical.



Closer to the Roche limit the body is deformed by tidal forces.



Within the Roche limit the mass's own gravity can no longer withstand the tidal forces, and the body disintegrates.



Particles closer to the primary move more quickly than particles farther away, as represented by the red arrows.

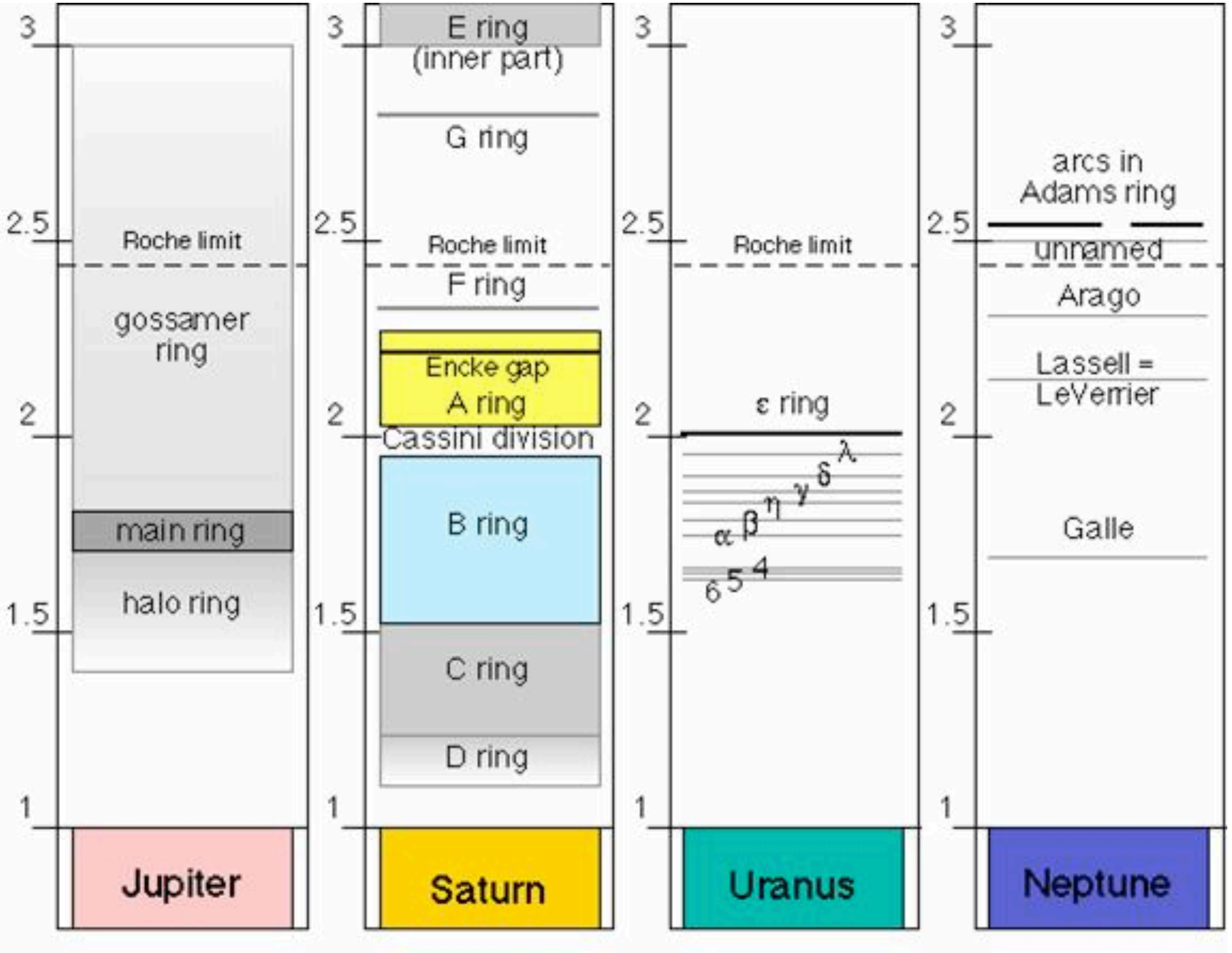
The varying orbital speed of the material eventually causes it to form a ring.

Don Hubbard

Roche Limit and rings

- Rings of giant planets are found mostly within (and a little bit outside) the Roche limit

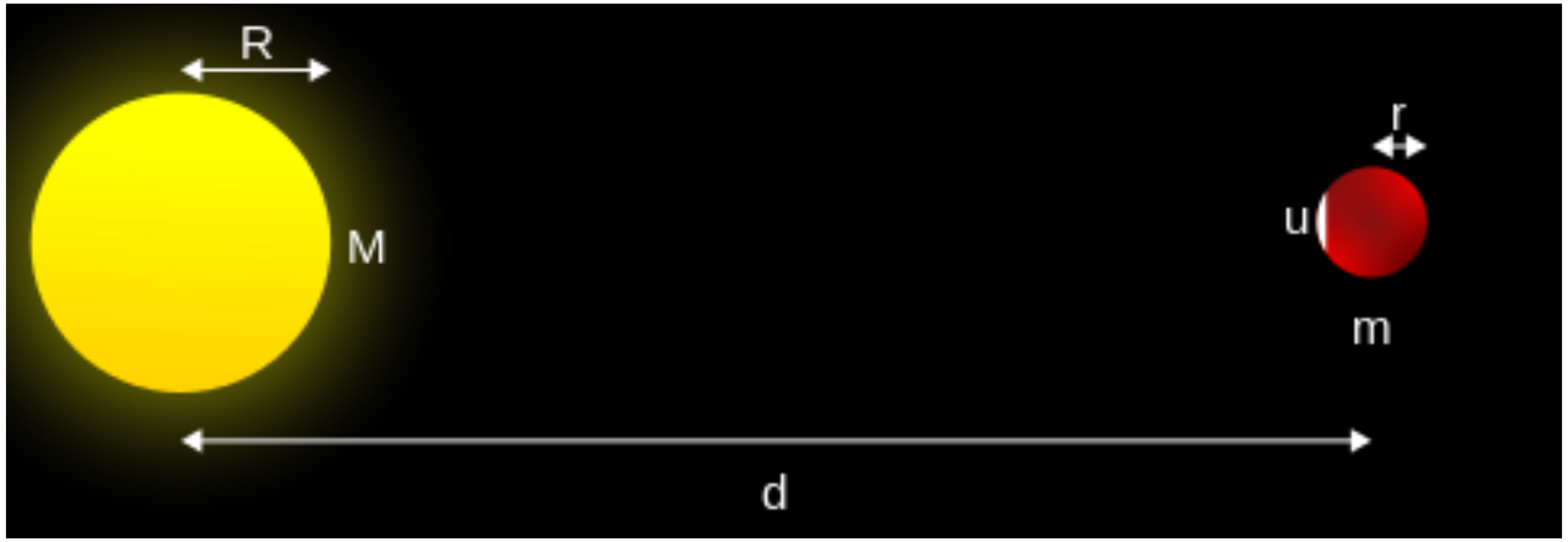
asterism.org



Rings of the jovian planets *scaled to size of the planet.* The distances are from the planet center in units of the planet radius. The classical Roche limit for a moon of equal density as the planet is shown as the dashed line at 2.44 planet radii.

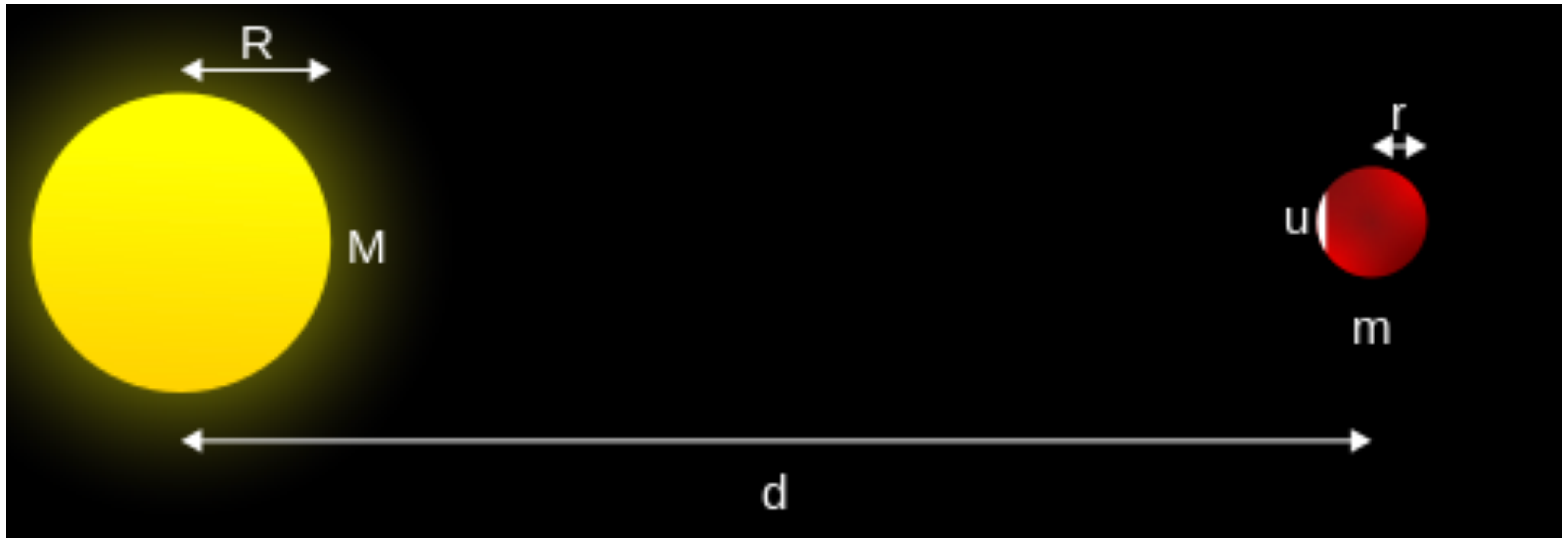
Roche Limit and Hill Radius

- Roche Limit: $d = r\left(\frac{2M}{m}\right)^{\frac{1}{3}}$
- Hill radius: $R_H = \left(\frac{M_2}{3(M_1 + M_2)}\right)^{\frac{1}{3}} a * (1 - e)$
- It's not a coincidence that they have a similar form: in both cases we find the radius where tidal force is large enough to pull something apart
  - Roche limit: tidal forces pull the moon apart
  - Hill sphere: tidal forces pull the moon+planet system apart



In-Class Activity
Order of Magnitude

- Hill radius: $R_H = \left(\frac{M_2}{3(M_1 + M_2)} \right)^{\frac{1}{3}} a * (1 - e)$
- Roche Limit: $d \approx 2.44R \left(\frac{\rho_M}{\rho_m} \right)^{\frac{1}{3}}$
- (1) What is the furthest Earth's moon could get from the Earth and still orbit the Earth, in Earth radii?
- (2) What is the closest Earth's moon could get to Earth in its orbit, in Earth radii?
- (3) What is the current semi-major axis of Earth's moon (hint: there are 28 days in a month), in Earth radii?



In-Class Activity
Order of Magnitude

- (1) Hill radius: $R_H = \left(\frac{M_2}{3(M_1+M_2)}\right)^{\frac{1}{3}} a*(1-e)$

Mass of Sun: 1 solar mass (M_1)
semi-major axis of Earth: 1 AU
1 Earth radius: $6x10^8$ cm

Mass of Earth: $3x10^{-6}$ solar mass (M_2)
1 AU = $1.5x10^{13}$ cm
eccentricity of Earth: 0.015 = 0

- Plugging in numbers:

$$R_H = \left(\frac{3x10^{-6}}{3(1)}\right)^{\frac{1}{3}}(1.5x10^{13}cm)\frac{1R_E}{6x10^8cm} = (10^{-6})^{\frac{1}{3}}\frac{1.5x10^{13}}{6x10^8}R_E = 10^{-2}\frac{1}{4}10^5R_E = 300R_E$$

- (2) Roche limit: $d \approx 2.44R\left(\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}}$

- Earth's moon is less dense than Earth, but for an order of magnitude: they're equally dense

$$d \approx 2.44R\left(\frac{\rho_M}{\rho_m}\right)^{\frac{1}{3}} = 2.44R_E(1)^{\frac{1}{3}} = 2.44R_E$$

In-Class Activity
Order of Magnitude

- (3) Kepler's third law: $P^2 = \frac{a^3}{M}$

From above: Mass of Earth: 3x10^{-6} solar mass P = 28 days = 0.1 years

- $a = (P^2 M)^{\frac{1}{3}} = ((0.1)^2(3x10^{-6}))^{\frac{1}{3}} * \frac{1.5x10^{13}cm}{6x10^8 cm} * R_E = (3x10^{-8})^{\frac{1}{3}} * \frac{1}{4} * 10^5 * R_E$

$= (30x10^{-9})^{\frac{1}{3}} * \frac{1}{4} * 10^5 * R_E = \frac{3}{4} * 10^{-3} * 10^5 R_E = 0.8x10^2 R_E = 80R_E$

For next time

- Reading: de Pater & Lissaeuer Chaper 2, section 2.7.2-2.7.4
- Homework 1 due tonight at 11:59pm on Canvas (reminder, late homework loses 10% of possible points each day)