1a) Escape velocity is given by: $v_{esc} = \sqrt{\frac{2GM}{r}}$

For a spherical object of constant density, $M = \rho * V = \rho * \frac{4\pi}{3}r^3$

Combining, $v_{esc} = \sqrt{\frac{2 * G * 4 * \rho * \pi * r^3}{3r}} = \sqrt{\frac{8\pi G\rho r^2}{3}}$

Solving for r, $r = v_{esc}\sqrt{\frac{3}{8\pi G\rho}}$

And plugging in numbers: $r = (150\frac{km}{hr} * \frac{10^5 cm}{1km} * \frac{1hr}{3600s}) * \sqrt{\frac{3}{8\pi(6.67x10^{-8}\frac{cm^3}{gs^2}) * (3\frac{g}{cm^3})}}$

$r = 3.22x10^6 cm = 32.2km$ (10 points)

1b) Energy conservation, per unit mass: $E = \frac{1}{2}v^2 - \frac{GM}{r} = constant$

Set energy at start of throw (1) to energy at highest point (2) equal:

$$\frac{1}{2}v_1^2 - \frac{GM}{r} = \frac{1}{2}v_2^2 - \frac{GM}{r+H}$$

At highest point, $v_2 = 0$, r is the radius of the asteroid, and H is the height above the surface

$$(r+H) * (\frac{1}{2}v_1^2 - \frac{GM}{r}) = -GM$$

As before, $M = \rho * V = \rho * \frac{4\pi}{3}r^3$

$$(r+H) * (\frac{1}{2}v_1^2 - \frac{4\pi\rho G r^2}{3}) = -\frac{4\pi\rho G r^3}{3}$$

$$H * \frac{1}{2}v_1^2 - H * \frac{4\pi\rho G r^2}{3} + r * \frac{1}{2}v_1^2 - \frac{4\pi\rho G r^3}{3} = -\frac{4\pi\rho G r^3}{3}$$

Cancel the r^3 term that appears on both sides of the equation:

$$H * \frac{1}{2}v_1^2 - H * \frac{4\pi\rho G r^2}{3} + r * \frac{1}{2}v_1^2 = 0$$

That’s a quadratic in r, so use the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where: } a = -\frac{4\pi\rho GH}{3}, b = \frac{1}{2}v_1^2, c = \frac{H}{2}v_1^2$$

1b) (continued)

Plugging in numbers: $a = -\frac{4}{3}\pi(3\frac{g}{cm^3})(6.67x10^{-8}\frac{cm^3}{gs^2})(50km * \frac{10^5cm}{1km}) = -4.19\frac{cm}{s^2}$

$$b = \frac{1}{2}(150\frac{km}{hr} * \frac{10^5cm}{km} * \frac{1hr}{3600s})^2 = 8.68 * 10^6\frac{cm^2}{s^2}$$

$$c = \frac{H}{2}v_1^2 = \frac{1}{2}(50km * \frac{10^5cm}{1km})(150\frac{km}{hr} * \frac{10^5cm}{km} * \frac{1hr}{3600s})^2 = 4.34x10^{13}\frac{cm^3}{s^2}$$

Plugging into the quadratic equation, get two solutions:

r = $-2.35x10^6$ cm (negative, so ignore), and a positive solution:

r = $4.42x10^6$ cm = 44.2 km (10 points)

1c) Stable circular orbit, at radius r, is given by

$$v_o = \sqrt{\frac{GM}{r}}$$

One more time, replace M with $M = \rho * V = \rho * \frac{4\pi}{3} r^3$

$$v_o = \sqrt{\frac{G * 4 * \rho * \pi * r^3}{3r}} = \sqrt{\frac{4\pi G \rho r^2}{3}}$$

And again, solve for r: $r = v_o \sqrt{\frac{3}{4\pi G \rho}}$

And plugging in numbers: $r = (150\frac{km}{hr} * \frac{10^5 cm}{1km} * \frac{1hr}{3600s}) * \sqrt{\frac{3}{4\pi(6.67x10^{-8}\frac{cm^3}{gs^2}) * (3\frac{g}{cm^3})}}$

$r = 4.55x10^6 cm = 45.5km$ (10 points)

2a) Hohmann transfer orbit between Jupiter and Earth

Earth semi-major axis: 1 AU
Jupiter semi-major axis: 5.2 AU

Perihelion: (1-e) * a
Aphelion: (1+e) * a

2a = Perihelion + Aphelion = 6.2 AU, a = 3.1 AU
2ae = Aphelion - Perihelion = 4.2 AU, so e = 4.2 / 2 / 3.1 = 0.67

We can use the vis-viva equation to get speed at perihelion:

$$v = \sqrt{GM_\odot\left(\frac{2}{r} - \frac{1}{a}\right)}$$

If r = 1 AU, and a = 3.1 AU:

$$v = \sqrt{(6.67x10^{-8}\frac{cm^3}{gs^2}) * (2x10^{33}g)\left(\frac{2}{(1AU * \frac{1.5x10^{13}cm}{1AU})} - \frac{1}{(3.1AU * \frac{1.5x10^{13}cm}{1AU})}\right)}$$

$$v = 3.86x10^6\frac{cm}{s} = 38.6\frac{km}{s}$$

Circular velocity of the Earth, as we get to assume e=0 (sorry Kepler):

$$v_c = \sqrt{\frac{GM_\odot}{d}} = \sqrt{\frac{(6.67x10^{-8}\frac{cm^3}{gs^2})(2x10^{33}g)}{(1.5x10^{13}cm)}} = 2.98x10^6\frac{cm}{s} = 29.8\frac{km}{s}$$

So in Earth's reference frame, the velocity needed is 38.6 - 29.8 = 8.8 km/s

(10 points)

2b) From the previous problem, we need to reach a velocity of 38.6 km/s

But we have a head start, since Earth's orbital velocity gets us pretty close to that.

So we need to have an initial velocity such that at infinity (from Earth), we reach a velocity that's the difference between 38.6 and Earth's orbital velocity

As before, start by conserving energy:

$$\frac{1}{2}v_1^2 - \frac{GM}{r_1} = \frac{1}{2}v_2^2 - \frac{GM}{r_2}$$

Set r_2 = infinity, so we can drop the last term:

$$\frac{1}{2}v_1^2 = \frac{GM}{r_1} + \frac{1}{2}v_2^2$$

So our target velocity is: $v_2 = (38.6\frac{km}{s} - v_c)$

Plugging in, and solving for initial velocity:

$$v_1 = \sqrt{2 * (\frac{GM}{r_1} + \frac{1}{2}(38.6\frac{km}{s} - \sqrt{\frac{GM_\odot}{d}})^2)}$$

Then, with M = mass of the Earth, r1 radius of the Earth, and d the Earth-Sun distance, plug in numbers:

$$v_1 = \sqrt{2 * (\frac{(6.67x10^{-8}\frac{cm^3}{gs^2})(5.97x10^{27}g)}{(6.37x10^8 cm)} + \frac{1}{2}(3.86x10^6\frac{cm}{s} - \sqrt{\frac{(6.67x10^{-8}\frac{cm^3}{gs^2})(2x10^{33}g)}{(1.5x10^{13}cm)}})^2)} = 1.42x10^6\frac{cm}{s} = 14.2\frac{km}{s} \quad \text{(10 points)}$$

2c) Very similar to the previous problem, but we get an additional velocity boost from Earth's rotation

Rotational velocity of the Earth at the equator:

$$v_r = \frac{2\pi r}{P_r} = \frac{2\pi(6.37x10^8 cm)}{(1day * \frac{24hr}{1day} * \frac{3600s}{1hr})} = 4.63x10^4 \frac{cm}{s} = 0.463 \frac{km}{s}$$

Our target velocity is still: $v_2 = (38.6\frac{km}{s} - v_c)$

But our starting velocity is now: $v_s = v_1 + v_r$

And we get a new velocity of:

$$v_1 = \sqrt{2 * (\frac{(6.67x10^{-8}\frac{cm^3}{gs^2})(5.97x10^{27}g)}{(6.37x10^8 cm)} + \frac{1}{2}(3.81x10^6 \frac{cm}{s} - \sqrt{\frac{(6.67x10^{-8}\frac{cm^3}{gs^2})(2x10^{33}g)}{(1.5x10^{13}cm)}})^2)} - (4.63x10^4 \frac{cm}{s}) = 1.39x10^6 \frac{cm}{s} = 13.8 \frac{km}{s} \text{ (10 points)}$$

So 460 m/s less…every little bit helps!

2d) We know from question 2a that the semi-major axis of the transfer orbit is 3.1 AU

Kepler's third law (in clever units) tells us that:

$$P^2 = \frac{a^3}{M}$$

We assume the spacecraft is much less massive than the Sun, and get an orbital period of:

$$P = \sqrt{\frac{(3.1)^3}{1}}$$ Since semi-major axis is in AU, and mass is in solar masses

P = 5.46 years

But, we're only doing the transfer orbit from periastron to apastron, which will be exactly half the orbit:

t = P/2 = 2.73 years (10 points)

3a)

Perihelion = (1-e) * a
Aphelion = (1+e) * a

If a = 3500 AU, and e = 0.99966:

Perihelion = (1 - 0.99966) * 3500 AU = 1.19 AU
Aphelion = (1 + 0.99966) * 3500 AU = 6998.81 AU

(10 points)

3b) This calls for the vis-viva equation

$$v = \sqrt{GM_{\odot}\left(\frac{2}{r} - \frac{1}{a}\right)}$$

If r = 1.19 AU, and a = 3500 AU:

$$v_p = \sqrt{(6.67x10^{-8}\frac{cm^3}{gs^2}) * (2x10^{33}g)\left(\frac{2}{(1.19AU * \frac{1.5x10^{13}cm}{1AU})} - \frac{1}{(3500AU * \frac{1.5x10^{13}cm}{1AU})}\right)}$$

$$v_p = 3.86578x10^6\frac{cm}{s} = 38.6578\frac{km}{s}$$

If r = 3998.81 AU, and a = 3500 AU:

$$v_a = \sqrt{(6.67x10^{-8}\frac{cm^3}{gs^2}) * (2x10^{33}g)\left(\frac{2}{(6998.81AU * \frac{1.5x10^{13}cm}{1AU})} - \frac{1}{(3500AU * \frac{1.5x10^{13}cm}{1AU})}\right)}$$

$$v_a = 657.28\frac{cm}{s} = 0.0065728\frac{km}{s}$$ (10 points)

3c)
Escape velocity is

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

At perihelion (1.19 AU) and Aphelion (6999 AU), escape velocity is:

$$v_{esc} = \sqrt{\frac{2(6.67x10^{-8}\frac{cm^2}{gs^2})(2x10^{33}g)}{(1.19 * 1.5x10^{13}cm)}} = 3.866106x10^6\frac{cm}{s} = 38.66106\frac{km}{s}$$

and:

$$v_{esc} = \sqrt{\frac{2(6.67x10^{-8}\frac{cm^2}{gs^2})(2x10^{33}g)}{(6999 * 1.5x10^{13}cm)}} = 5.0411x10^4\frac{cm}{s} = 0.50411\frac{km}{s}$$

So at periastron, change in velocity is: (38.66106 - 38.6578) km/s = 0.0033 km/s

And at apastron: (0.504 - 0.00603) km/s = 0.498 km/s

So it’s easier to eject at periastron than apastron. (10 points)